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SEQUENTIAL CONSENSUS FOR SELECTING QUALIFIED INDIVIDUALS OF A GROUP

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In this paper we analyze a group decision procedure that follows a recursive pattern. In the first stage, the members of a group show their opinions on all the individuals of that group, regarding a specific attribute, by means of assessments within an ordered set, e.g. the unit interval or a finite scale. Taking into account this information, some aggregation operators and a family of thresholds, a subgroup of individuals is selected: those members whose collective assessment reach a specific threshold. Now only the opinions of this qualified subgroup are taken into account and a new subgroup emerges in the implementation of the aggregation phase. We analyze how to put in practice this recursive procedure in order to provide a final subgroup of qualified members. We have considered the minimum as aggregation operator. Thus, the collective assessment is just the worst of the individual assessments. This idea corresponds to qualify individuals whenever all the individual assessments reach the fixed threshold.

Keywords: Group decision making; aggregation operators; consensus; qualification.

1. Introduction

In some occasions a group of individuals has to select a subgroup for doing a task or regarding an attribute. This is the case of a group of experts that has to decide which members should participate in a concrete task. Another possibility consists on determining which members of a society possess a specific characteristic or ability. In the remaining, we refer to such subgroup as qualified members.

There exist in the literature some papers where the problem arises in choosing the members of the society satisfying a social identity (see Kasher and Rubinstein [1] and Dimitrov, Sung and Xu [2], among others), or with respect to a general attribute (see Samet and Schmeidler [3]).

Aggregation operators allow us to generate a collective assessment to each individual taking into account the individual opinions (see Fodor and Roubens [4], Grabisch, Orlovski and Yager [5] and Calvo, Kolesárova, Komorníková and Mesiar [6], among others). A simple way of selecting qualified members can be done by means of thresholds: all the individuals for which collective assessment reaches the threshold are considered as qualified.

However, typically qualified members have more valuable opinions about the task or topic under discussion. Hence, the final set of qualified members should possess the intuitive property of being a *stable subgroup*, i.e., a group such when considering their assessments on all society members, they would select themselves and only themselves as qualified. Unfortunately, no aggregation operator can generally guarantee this.

This appealing idea can be obtained through a sequential procedure. The society selects a subgroup combining an aggregation operator and a threshold. Once this is done, the members of this subgroup selects a new subgroup combining their opinions (but only theirs) and a threshold. Recursively, new subgroups appear, and if the sequence of subgroups is convergent, a stable set of qualified members arises.

It is fundamental to determine which operators behave well, in the sense that for all possible opinions of individuals, the sequential process ends up in a stable subgroup.

Among the large variety of aggregation operators that we can use for our sequential procedure, some classical weighting operators have been analyzed in Ballester and García-Lapresta [7]. The case of OWA ("Ordered Weighted Averaging") operators (Yager [8]) has been widely discussed in Ballester and García-Lapresta [9] and in Aguiló *et al.* [10] where we extend the analysis to some classes of extended OWA operators generated by sequences and fractals. Moreover, in Ballester and García-Lapresta [11] we pay attention to the case of some classes of quasiarithmetic means.

To motivate this paper, let us first stress the relevance of the minimum aggregator as a way of conforming collective opinions. This operator captures the essence of the consensus idea. An individual is qualified if all the opinions about her are positive. Despite its relevance, in Ballester and García-Lapresta [9] we show that this operator is not convergent. There exist opinions within the society for which no stable subgroup arises from the sequential procedure.

This paper deeply analyzes the validity of the minimum operator to generate stable subgroups. To do that, we first characterize for which opinions within society stable qualified subgroups can be obtained. Whenever convergence is not possible, opinions within the society present a very specific cyclical structure. Second, making use of this structure, we can describe an intuitive way of solving the problem for cases in which stable groups do not appear.

The paper is organized as follows. Section 2 includes the main concepts we use in the paper. Section 3 is devoted to analyze the convergence of sequences generated by the minimum operator. In Section 4 we propose some procedures for dealing with non-convergent sequences. Finally, Section 5 includes some conclusions.

2. Preliminaries

Consider a finite set of individuals $N = \{1, 2, \ldots, n\}$ with $n \geq 2$. We use 2^N to denote the power set of N, i.e., the set of all the subsets of N, and $|S|$ is the cardinal of any subset S. A *profile* is an $n \times n$ matrix

$$P = \begin{pmatrix} p_{11} & \cdots & p_{1j} & \cdots & p_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{i1} & \cdots & p_{ij} & \cdots & p_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{n1} & \cdots & p_{nj} & \cdots & p_{nn} \end{pmatrix}$$

with values in a linear order $\langle L, < \rangle$ with maximum $\max L$ and minimum $\min L$, where p_{ij} is the assessment with which the individual i evaluates individual j as being qualified to belong to the committee in question.[1] The set of profiles is denoted by $\mathcal{P}$. Given a subset of individuals $S \subseteq N$, P_S denotes the $|S| \times n$ submatrix of P composed by those $i-$rows with $i \in S$. Given $j \in N$, we denote by P_S^j the j-th column vector of P_S.

The election of the linear order $\langle L, < \rangle$ where individuals show their opinions allows us to deal with different approaches, mainly:

(i) *Dichotomous assessments* whenever $L = \{0, 1\}$ and $0 < 1$. In this case, $p_{ij} = 0$ means that individual i thinks that individual j has not to be qualified, and $p_{ij} = 1$ means that individual i thinks that individual j has to be qualified.

(ii) *Gradual assessments* if $|L| > 2$. Now p_{ij} shows the degree with which individual i thinks that individual j is qualified with respect to the considered attribute.

   (a) *Fuzzy assessments* whenever $L = [0, 1]$ and $<$ is the usual order in the real line.

   (b) *Linguistic assessments* if $L = \{l_1, \ldots, l_g\}$ is a finite scale with granularity $|L| = g \geq 3$ and $l_1 < \cdots < l_g$. Now terms of L can be defined by means of linguistic labels as *very bad*, *bad*, *regular*, *good* and *very good*.

Definition 1. The *minimum evaluation function*

$$w : \mathcal{P} \times \left(2^N \setminus \{\emptyset\}\right) \times N \longrightarrow L$$

is defined by

$$w(P, S, j) = \min\{p_{ij} \mid i \in S\}.$$

So, given a profile $P \in \mathcal{P}$, a subgroup $\emptyset \neq S \subseteq N$ and an individual $j \in N$, $w(P, S, j) \in L$ is the *worst* individual assessment on j by the members of S in the profile P.

[1]In Ballester and García-Lapresta [9,11] and Aguiló, Ballester, Calvo, García-Lapresta, Mayor and Suñer [10] we only work with values in $[0, 1]$.

Remark 1. The minimum evaluation function satisfies the following properties:

(i) *Independence*: $w(P,S,j) = w(Q,S,j)$ for all $P,Q \in \mathcal{P}$ satisfying $P^j_S = Q^j_S$. Then, the collective assessment that the subgroup S provides to individual j, $w(P,S,j)$, only depends on the individual assessments of S on the individual j.
(ii) *Monotonicity*: $S \subseteq T \;\Rightarrow\; w(P,S,j) \geq w(P,T,j)$. In other words, when a subgroup of individuals decreases (increases), the collective assessment does not decrease (increase).

The minimum evaluation function determines individual qualification only in gradual terms. A very natural way to convert a gradual opinion into a dichotomic assessment is by means of thresholds. An individual is qualified if the collective assessment is above a fixed threshold. In other words, an individual is considered able by a certain group if all the members of the group agree on that point.

Definition 2. Given a threshold $\alpha \in L$, the *minimum qualification function associated with* α

$$W_\alpha : \mathcal{P} \times 2^N \longrightarrow 2^N$$

is defined by

$$W_\alpha(P,S) = \begin{cases} \{j \in N \mid w(P,S,j) \geq \alpha\}, & \text{if } S \neq \emptyset, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Clearly,

$$W_\alpha(P,S) = \{j \in N \mid \forall i \in S \quad p_{ij} \geq \alpha\}$$

for every nonempty subgroup $S \subseteq N$.

Remark 2. The minimum qualification function satisfies the following properties for every threshold $\alpha \in L$:

(i) *Independence*: $W_\alpha(P,S) = W_\alpha(Q,S)$ for all $P,Q \in \mathcal{P}$ satisfying $P_S = Q_S$. Then, the set of qualified members provided by the subgroup S, $W_\alpha(P,S)$, only depends on the individual assessments of S.
(ii) *Monotonicity*: $S \subseteq T \;\Rightarrow\; W_\alpha(P,T) \subseteq W_\alpha(P,S)$. In other words, if an individual has been qualified by the big subgroup, then this individual will be qualified by the small subgroup; equivalently, if an individual has not been qualified by the small subgroup, then this individual will not be qualified by the big subgroup.

3. Convergence

In this section, we introduce two notions of convergence and we characterize the set of profiles where convergence is not reached.

Definition 3. Given a threshold $\alpha \in L$ and a profile $P \in \mathcal{P}$, the sequence $\{S_t(\alpha, P)\}$ is defined by $S_1(\alpha, P) = N$ and $S_{t+1}(\alpha, P) = W_\alpha(P, S_t(\alpha, P))$.

(i) The sequence $\{S_t(\alpha, P)\}$ is *convergent* if there exists $q \in \mathbb{N}$ such that $S_q(\alpha, P) = S_{q+1}(\alpha, P) = \lim S_t(\alpha, P)$.
(ii) The qualification function W_α is *convergent* if the sequence $\{S_t(\alpha, P)\}$ is convergent for every $P \in \mathcal{P}$.

Notice that if $\alpha = \min L$, then $\lim S_t(\alpha, P) = N$ for every $P \in \mathcal{P}$.

Fig. 1 shows in a nutshell the main components of a sequential process of evaluation.

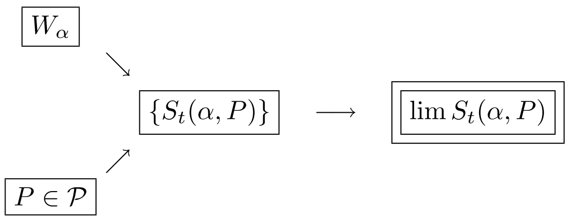

Fig. 1. Iterative procedure.

In Ballester and García-Lapresta [9] we show that for $L = [0, 1]$, the qualification function W_α is convergent if and only if $\alpha = \min L = 0$. For the sake of illustration, let us show with a particular example how convergence might not appear whenever $\alpha \in L \setminus \{\min L\}$.

Example 1. Consider a group of two agents $N = \{1, 2\}$ and the profile P defined by:

$$p_{ij} = \begin{cases} \max L, & \text{if } i = 1 \text{ or } j = 1, \\ \min L, & \text{otherwise.} \end{cases}$$

For every threshold $\alpha \in L \setminus \{\min L\}$, any subgroup would qualify agent 1. However, agent 2 is critical with herself, though the other agent considers her fully able. Thus, whenever the opinion of agent 2 is relevant, 2 is not qualified. Hence, it is easy to see that the sequential procedure determines:

$$S_t(\alpha, P) = \begin{cases} \{1, 2\}, & \text{if } t \text{ is odd,} \\ \{1\}, & \text{if } t \text{ is even.} \end{cases}$$

Thus, the sequence is not convergent.

With the following example we show how not only opinions of one individual about herself are fundamental for convergence. There might be also the case that two individuals eliminate themselves though being considered able by the rest of society.

Example 2. Consider a group of three agents $N = \{1,2,3\}$ and the profile P defined by:

$$p_{ij} = \begin{cases} \min L, & \text{if } (i=2 \text{ and } j=3) \text{ or } (i=3 \text{ and } j=2), \\ \max L, & \text{otherwise.} \end{cases}$$

For every threshold $\alpha \in L \setminus \{\min L\}$, agent 1 is considered as qualified by any subgroup. However, agent 2 is negative about agent 3, and viceversa. Thus, whenever both agents' opinions are relevant, agents 2 and 3 are not qualified.

It is easy to see that sequences for the considered profile are as follows:

$$S_t(\alpha, P) = \begin{cases} \{1,2,3\}, & \text{if } t \text{ is odd,} \\ \{1\}, & \text{if } t \text{ is even.} \end{cases}$$

Thus, the sequence is not convergent.

The previous examples illustrate how cycles appear in non-convergent sequences when we use the minimum operator and $\alpha \in L \setminus \{\min L\}$. Moreover, these examples describe specific characteristics of the profiles for which this convergence is violated. In the following result we characterize the set of profiles $P \in \mathcal{P}$ where the sequence $\{S_t(\alpha, P)\}$ is not convergent. The following lemmas are useful.

Lemma 1. *For every threshold $\alpha \in L \setminus \{\min L\}$ and every $P \in \mathcal{P}$, the odd subsequence $\{S_{2m-1}(\alpha, P) \mid m \in \mathbb{N}\}$ is decreasing and the even subsequence $\{S_{2m}(\alpha, P) \mid m \in \mathbb{N}\}$ is increasing.*

Proof. We start by proving that $\{S_{2m-1}(\alpha, P) \mid m \in \mathbb{N}\}$ is decreasing by using an induction argument. Since $S_1(\alpha, P) = N$, it is clear that $S_1(\alpha, P) \supseteq S_3(\alpha, P)$. Now suppose that the subsequence is decreasing from $m = 1$ to $m = M$, i.e.,

$$S_1(\alpha, P) \supseteq S_3(\alpha, P) \supseteq \cdots \supseteq S_{2M-3}(\alpha, P) \supseteq S_{2M-1}(\alpha, P).$$

We now justify $S_{2M-1}(\alpha, P) \supseteq S_{2M+1}(\alpha, P)$. From $S_{2M-3}(\alpha, P) \supseteq S_{2M-1}(\alpha, P)$, by monotonicity we have

$$S_{2M-2}(\alpha, P) = W_\alpha(P, S_{2M-3}(\alpha, P)) \subseteq W_\alpha(P, S_{2M-1}(\alpha, P)) = S_{2M}(\alpha, P)$$

and

$$S_{2M-1}(\alpha, P) = W_\alpha(P, S_{2M-2}(\alpha, P)) \supseteq W_\alpha(P, S_{2M}(\alpha, P)) = S_{2M+1}(\alpha, P).$$

A similar reasoning proves that $\{S_{2m}(\alpha, P) \mid m \in \mathbb{N}\}$ is increasing. □

Lemma 2. *For every threshold $\alpha \in L \setminus \{\min L\}$, every $P \in \mathcal{P}$ and all $m, m' \in \mathbb{N}$, it holds $S_{2m'}(\alpha, P) \subseteq S_{2m-1}(\alpha, P)$.*

Proof. We first prove that $S_{2m}(\alpha, P) \subseteq S_{2m-1}(\alpha, P)$ for every $m \in \mathbb{N}$. This is obviously true for $m = 1$. Now suppose this is true from $m = 1$ to $m = M$, and we justify this property for $m = M + 1$. Since $S_{2M}(\alpha, P) \subseteq S_{2M-1}(\alpha, P)$, monotonicity guarantees that

$$S_{2M+1}(\alpha, P) = W_\alpha(P, S_{2M}(\alpha, P)) \supseteq W_\alpha(P, S_{2M-1}(\alpha, P)) = S_{2M}(\alpha, P).$$

But applying monotonicity again over $S_{2M+1}(\alpha, P) \supseteq S_{2M}(\alpha, P)$, we have

$$S_{2M+2}(\alpha, P) = W_\alpha(P, S_{2M+1}(\alpha, P)) \subseteq W_\alpha(P, S_{2M}(\alpha, P)) = S_{2M+1}(\alpha, P).$$

This concludes the induction argument. Now, using Lemma 1, it is not difficult to prove the statement. Suppose, by contradiction, that the statement is not true. In this case, there would exist $m, m' \in \mathbb{N}$ such that $S_{2m'}(\alpha, P) \not\subseteq S_{2m-1}(\alpha, P)$. If $m' < m$, by Lemma 1 we have $S_{2m'}(\alpha, P) \subseteq S_{2m}(\alpha, P)$. Therefore, it must be $S_{2m}(\alpha, P)\} \not\subseteq S_{2m-1}(\alpha, P)$, a contradiction. Then it must be $m' \geq m$. But hence, Lemma 1 guarantees $S_{2m'-1}(\alpha, P) \subseteq S_{2m-1}(\alpha, P)$ and therefore it must be $S_{2m'}(\alpha, P) \not\subseteq S_{2m'-1}(\alpha, P)$, a contradiction. □

Remark 3. By Lemma 1, it is clear that for every threshold $\alpha \in L \setminus \{\min L\}$ and every $P \in \mathcal{P}$, the odd subsequence $\{S_{2m-1}(\alpha, P) \mid m \in \mathbb{N}\}$ and the even subsequence $\{S_{2m}(\alpha, P) \mid m \in \mathbb{N}\}$ are convergent. Moreover, as a consequence of Lemma 2, it is obvious that $\lim S_{2m}(\alpha, P) \subseteq \lim S_{2m-1}(\alpha, P)$. Consequently, convergence is only obtained whenever $\lim S_{2m}(\alpha, P) = \lim S_{2m-1}(\alpha, P)$.

When a sequence $\{S_t(\alpha, P)\}$ is not convergent for a profile $P \in \mathcal{P}$, there exists some agents in $\lim S_{2m-1}(\alpha, P)$ who are not in $\lim S_{2m}(\alpha, P)$. Intuitively, these particular agents are accepted by all agents in $\lim S_{2m-1}(\alpha, P)$, but once they are part of the set of qualified members, they expel themselves from the set of qualified members. The following theorem fully describes the structure of preferences among agents that lead to non-convergent situations.

Theorem 1. *Given a threshold $\alpha \in L \setminus \{\min L\}$, the sequence $\{S_t(\alpha, P)\}$ is not convergent if and only if there exists a partition of N, $\{A_k\}_{k=1}^r$, such that for all $p, q \in \{1, 2, \ldots, r\}$:*

(i) Whenever r is even,

(a) If $p + q \leq r$ or $(p + q = r + 1$ and $p \geq q - 1)$:

$$\forall i \in A_p \quad \forall j \in A_q \quad p_{ij} \geq \alpha.$$

(b) If $(p + q = r + 1$ and $p < q - 1)$ or $(p + q = r + 2$ and $p \geq q)$:

$$\forall j \in A_q \quad \exists i \in A_p \quad p_{ij} < \alpha.$$

(ii) Whenever r is odd,

(a) If $p + q \leq r$ or $(p + q = r + 1$ and $p \geq q + 2)$:

$$\forall i \in A_p \quad \forall j \in A_q \quad p_{ij} \geq \alpha.$$

(b) If $(p+q=r+1 \text{ and } p<q+2)$ *or* $(p+q=r+2 \text{ and } p\geq q+3)$*:*

$$\exists i \in A_p \quad \exists j \in A_q \quad p_{ij} < \alpha.$$

Proof. We first prove the sufficiency part. Consider a profile P and a partition of the set of agents, $\{A_k\}_{k=1}^r$, with the characteristics described by the theorem. Let us consider the case in which r is even (the other part is similar and thus omitted). A detailed inspection of the profile guarantees that the sequence of subgroups is given by:

$$S_1(\alpha,P)=N,\; S_{2m}(\alpha,P)=\bigcup_{k=1}^{m} A_k,\; S_{2m+1}(\alpha,P)=\bigcup_{k=1}^{r-m} A_k,\; m=1,2,\ldots,\frac{r-2}{2}.$$

Consider now the set $S_{r-1}(\alpha,P)$. Given the characteristics of the profile, this set does not accept new members and excludes from the group to agents in $A_{\frac{r+2}{2}}$. Thus, $S_{r-1}=\bigcup_{k=1}^{\frac{r}{2}} A_k$. However, all the agents in this group accept again to all the members of $A_{\frac{r+2}{2}}$ and only to these, generating a cycle and avoiding convergence.

We now prove the necessary part. Consider now the first value s such that $S_s(\alpha,P)=S_p(\alpha,P)$ for some $p<s$. Given that the sequence is not convergent, it cannot be $p=s-1$. Given Lemma 2, it can be only $p=s-2$. Suppose that s is even (the case of s being odd is similar and thus omitted). We now construct a partition of the set of agents with the characteristics described by the theorem. Denote by B_0 the empty set and by $B_1,B_2,\ldots,B_s$ the s sets $S_0,S_1,\ldots,S_{s-1}$ ordered in a way such that $B_k\subseteq B_{k+1}$ for every $k\in\{1,2,\ldots,s-1\}$. Now define $A_k=B_{k+1}\setminus B_k$ for $k=1,2,\ldots,r=s-1$. It is a mechanical task to check that these sets should verify the conditions of the theorem. □

Theorem 1 shows how the absence of convergence is due to a particular structure of opinions. In the last section we make use of this structure to describe an intuitive solution whenever convergence is not possible.

4. Dealing with non-convergent sequences

Let us reconsider how cycles appear in non-convergent sequences. The odd subsequence $\{S_{2m-1}(\alpha,P) \mid m\in\mathbb{N}\}$ is decreasing, whereas the even subsequence $\{S_{2m}(\alpha,P) \mid m\in\mathbb{N}\}$ is increasing. If the profile has the characteristics described in Theorem 1, we have $\lim S_{2m}(\alpha,P) \subsetneq \lim S_{2m-1}(\alpha,P)$. Moreover, for the case of r even, we obtain:

$$\lim S_{2m-1}(\alpha,P)=\bigcup_{k=1}^{\frac{r+2}{2}} A_k,$$

but

$$\lim S_{2m}(\alpha, P) = \bigcup_{k=1}^{\frac{r}{2}} A_k.$$

It seems quite intuitive that the solution to the problem should be some subset of $\lim S_{2m-1}(\alpha, P)$ and a superset of $\lim S_{2m}(\alpha, P)$. The evaluation of agents in $\lim S_{2m-1}(\alpha, P)$ does not seem to be arguable. For instance, if agents in $\lim S_{2m-1}(\alpha, P)$ would modify their opinions about agents in $A_{\frac{r+2}{2}}$, excluding them from being qualified, agents in $\lim S_{2m-1}(\alpha, P)$ would be the final stable subgroup of qualified members. Agents not in $\lim S_{2m}(\alpha, P)$ should not be considered in any intuitive solution to the problem, since all of them are excluded from completely valid agents (those in $\lim S_{2m-1}(\alpha, P)$). Hence, any debate should be focused about what to do with conflictive individuals in $A_{\frac{r+2}{2}}$.

Our suggestion to solve this question is to eliminate conflictive individuals in $A_{\frac{r+2}{2}}$ from the final subgroup of qualified members. Given that our fundamental interest is to obtain consensus on the set of qualified members and these conflictive agents are not able to reach an agreement about their validity, they could be excluded. This seems to be the most prudent decision, avoiding conflicts inside the committee.

Summarizing, our proposal is to take $\lim S_{2m}(\alpha, P)$ as final set of qualified individuals.

Example 3. Consider 10 decision makers, $N = \{1, \ldots, 10\}$, who show their opinions according to the set of linguistic labels $L = \{l_1, l_2, l_3, l_4, l_5\}$ given in Table 1.

Table 1. Linguistic labels.

$\mathcal{L}$	Meaning
l_1	Very bad
l_2	Bad
l_3	Regular
l_4	Good
l_5	Very good

The opinions of these individuals are included in the following profile:

$$P = \begin{pmatrix} l_5 & l_4 & l_4 & l_4 & l_5 & l_2 & l_4 & l_5 & l_4 & l_1 \\ l_5 & l_3 & l_2 & l_5 & l_5 & l_3 & l_4 & l_5 & l_4 & l_2 \\ l_5 & l_4 & l_3 & l_5 & l_5 & l_3 & l_3 & l_5 & l_5 & l_3 \\ l_5 & l_4 & l_3 & l_4 & l_5 & l_2 & l_5 & l_5 & l_5 & l_1 \\ l_5 & l_5 & l_5 & l_4 & l_5 & l_3 & l_4 & l_4 & l_4 & l_2 \\ l_5 & l_3 & l_3 & l_3 & l_5 & l_4 & l_5 & l_5 & l_2 & l_3 \\ l_5 & l_3 & l_2 & l_4 & l_5 & l_3 & l_3 & l_5 & l_5 & l_2 \\ l_5 & l_4 & l_4 & l_5 & l_5 & l_3 & l_4 & l_5 & l_4 & l_3 \\ l_5 & l_5 & l_3 & l_4 & l_5 & l_4 & l_5 & l_5 & l_5 & l_3 \\ l_4 & l_3 & l_1 & l_2 & l_5 & l_4 & l_3 & l_5 & l_3 & l_4 \end{pmatrix}.$$

(i) If we consider the threshold $\alpha = l_3$, the sequence $\{S_t(l_3, P)\}$ is convergent:

$$S_1(l_3, P) = N$$
$$S_2(l_3, P) = \{1, 2, 5, 7, 8\}$$
$$S_{m+2}(l_3, P) = \{1, 2, 4, 5, 7, 8, 9\}, \text{ for every } m \in \mathbb{N}.$$

Consequently, $\lim S_t(l_3, P) = \{1, 2, 4, 5, 7, 8, 9\}$.

(ii) If we consider the threshold $\alpha = l_4$, we obtain the following subsets of qualified individuals:

$$S_1(l_4, P) = N$$
$$S_2(l_4, P) = \{1, 5, 8\}$$
$$S_3(l_4, P) = \{1, 2, 3, 4, 5, 7, 8, 9\}$$
$$S_4(l_4, P) = \{1, 4, 5, 8, 9\}$$
$$S_5(l_4, P) = \{1, 2, 4, 5, 7, 8, 9\}$$
$$S_6(l_4, P) = \{1, 4, 5, 8, 9\}$$
$$S_7(l_4, P) = \{1, 2, 4, 5, 7, 8, 9\}$$
$$S_8(l_4, P) = \{1, 4, 5, 8, 9\}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$

Clearly, this sequence does not converge because there exists a cycle: $S_{2m+3}(l_4, P) = \{1, 2, 4, 5, 7, 8, 9\}$ and $S_{2m+4}(l_4, P) = \{1, 4, 5, 8, 9\}$, for every $m \in \mathbb{N}$.

In order to obtain the final set of qualified individuals, let us first describe the partition of N, $\{A_k\}_{k=1}^r$ that appears with this profile, as announced by Theorem 1. Clearly, $r = 5$ and

$$A_1 = \{1, 5, 8\},\ A_2 = \{4, 9\},\ A_3 = \{2, 7\},\ A_4 = \{3\},\ A_5 = \{6, 10\}.$$

Conflictive agents belong to group A_3. They are part of the limit set of the decreasing (odd) sequence, but not from the increasing (even) sequence. This is the case because all agents in A_1 and A_2 consider agents 2 and 7 valid. However, agent 7 considers that both agent 2 and herself are not valid. Similarly, agent 2 considers herself not valid. These agents could be eliminated from the set of qualified members and thus, a final subgroup should be the limit of the increasing sequence, i.e., $A_1 \cup A_2$.

Thus, the final set of qualified individuals is $\lim S_{2m}(l_4, P) = \{1, 4, 5, 8, 9\}$.

(iii) If we consider the threshold $\alpha = l_5$, the sequence $\{S_t(l_5, P)\}$ is convergent:

$$\begin{aligned} &S_1(l_5, P) = N \\ &S_2(l_5, P) = \{5\} \\ &S_3(l_5, P) = \{1, 2, 3, 5\} \\ &S_{m+3}(l_5, P) = \{1, 5\}, \text{ for every } m \in \mathbb{N}. \end{aligned}$$

Consequently, $\lim S_t(l_5, P) = \{1, 5\}$.

5. Concluding remarks

In this paper we have considered a recursive decision procedure where individuals show their opinions on all the members of the society with respect to an attribute. These assessments should be provided within a linear order with maximum and minimum. This informational framework generalizes those of dichotomous and fuzzy assessments and allows us to work with finite scales made up by linguistic labels, as usual in real life.

It is worth emphasizing that individuals only show their opinions once, and the recursive procedure sequentially generates a new group of qualified individuals taking into account only the opinions of the qualified individuals in the previous stage.

In order to achieve maximum consensus, we consider that the collective assessment is just the worst individual opinion. So, if the collective assessment is a given term, then this means that all the individuals agree that the evaluated member of the group reach this degree of qualification.

The sequential procedure is defined as follows. Given a constant threshold belonging to the linear order, we initially select those individuals whose collective assessments reach that threshold. Secondly, we only consider the opinions of this set of qualified individuals for generating a new set of qualified members: those individuals of the society whose collective assessments reach that threshold. We repeat this process until we obtain a final set of qualified members of the society. We have analyzed the convergence of this sequential procedure and we have proposed how to define an undeniable set of qualified individuals.

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