Extension of some Voting Systems to the Field of Gradual Preferences

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Summary. In the classical theory of social choice, there exist many voting procedures for determining a collective preference on a set of alternatives. The simplest situation happens when a group of individuals has to choose between two alternatives. In this context, some voting procedures such as simple and absolute special majorities are frequently used. However, these voting procedures do not take into account the intensity with which individuals prefer one alternative to the other. In order to consider this situation, one possibility is to allow individuals showing their preferences through values located between 0 and 1. In this case, the collective preference can be obtained by means of an aggregation operator. One of the most important matter in this context is how to choose such aggregation operator. When we consider the class of OWA operators, it is necessary to determine the associated weights. In this contribution we survey several methods for obtaining the OWA operator weights. We pay special attention to the way the weights are chosen, regarding the concrete voting system we want to obtain when individuals do not grade their preferences between the alternatives.

1 Introduction

When a group of agents have to choose between two alternatives taking into account the individual opinions, there exist two main features for reaching this decision: how the agents show their preferences and how to aggregate the information they provide. With respect to the first aspect, individuals can declare their opinions in a dichotomous manner by showing which alternative is the best or by declaring indifference between these alternatives (this is the common way in classic voting systems); another possibility consists on allowing individuals to show gradual preferences in someway. In this sense, valued and fuzzy preferences consider numerical values for declaring intensities of preference (see Nurmi [57], Tanino [63], Fodor and Roubens [25], De Baets and Fodor [12], García-Lapresta and Llamazares [29, 30], Llamazares and García-Lapresta [44], Llamazares [41, 43] and Fodor and De Baets [23], among others). After Zadeh [83], linguistic preferences have been very used in the

Decision Theory framework (see Delgado *et al.* [14, 15], Yager [76], Herrera *et al.* [36, 37, 38], Bordogna *et al.* [5], Herrera and Herrera-Viedma [35]) and in voting systems (see García-Lapresta [28] and García-Lapresta *et al.* [31]).

It is worth emphasizing that the outcome of a vote depends not only on the aggregation procedure, but on the way the individuals show their opinions. In fact, individuals usually feel different intensities of preference when they compare pairs of alternatives. However, in the classic voting procedures they can not declare these intensities, and different modalities of preference and identified in a unique way —for instance, as if they feel extreme preference. So, individual opinions are truncated and misrepresented. In order to avoid this drawback, an interesting problem is to extend classic voting systems in such a way that they could aggregate intensities of preference. A possibility is to consider the following procedure: Once individuals show their preferences through a value between 0 and 1, we obtain the collective intensity of preference by means of an OWA operator. From this value and through a kind of strong α-cut, where $\alpha \in [\frac{1}{2}, 1)$, we can decide if an alternative is chosen or if both alternatives are collectively indifferent.

When individuals do not grade their preferences, the previous procedure allows us to obtain a voting system in the classic way. Hence, once fixed α, it is possible to know what class of voting systems underlies in the aggregation process according to the used OWA operator. In this sense, we show that we can determine —by means of the aforementioned procedure— the OWA operators which generalize simple, absolute, Pareto, unanimous and absolute special majorities. Moreover, because there exist multiple OWA operators generating a specific classic voting system, we also propose a way for choosing the best-suited OWA operator. On the other hand, it is worth emphasizing that the induced extensions maintain some good properties of the genuine voting procedures such as symmetry, self-duality, monotonicity and unanimity.

Although in this chapter we focus on OWA operators, it is worth emphasizing that the previous procedure has been already used to characterize some classes of aggregation operators which extend some voting systems. So, García-Lapresta and Llamazares [30] generalize two classes of majorities based on difference of votes by using quasiarithmetic means and window (olympic) OWA operators as aggregation operators.

The chapter is organized as follows. Section 2 is devoted to crisp aggregation operators, their properties and some classic voting systems that can be defined with them. Moreover, we point out some drawbacks of some usual voting systems. In Section 3, we deal with general aggregation operators and particularly with OWAs. We also survey some methods appeared in the literature in order to determine the OWA operator weights. In Section 4, we extend several classic voting systems through OWA operators in the case that individuals show intensities of preference by means of numerical values within the unit interval, and we provide some characterization results. Moreover, we propose a method for choosing the best-suited OWA operators which extend some voting systems. We also obtain the crisp aggregation operators associated with these best-suited OWAs. Finally, some conclusions are included in Section 5.

2 Crisp aggregation operators

The simplest situation in the collective decision making procedures happens when a group of individuals has to choose between two alternatives x and y. When individuals do not grade their preferences, some authors, such as May [49] and Fishburn [21], have used an index d in order to distinguish among the three possible cases of ordinary preference and indifference between x and y:

$$d = \begin{cases} 1, & \text{if } x \text{ is preferred to } y, \\ 0, & \text{if } x \text{ is indifferent to } y, \\ -1, & \text{if } y \text{ is preferred to } x. \end{cases}$$

In order to extend classical voting systems to the field of gradual preferences we define an equivalent index $r = (d+1)/2$. In this way, we have:

$$r = \begin{cases} 1, & \text{if } x \text{ is preferred to } y, \\ \frac{1}{2}, & \text{if } x \text{ is indifferent to } y, \\ 0, & \text{if } y \text{ is preferred to } x. \end{cases}$$

We consider m voters, $m \geq 2$, who show their preferences between x and y. A *crisp profile* is a vector $\boldsymbol{r} = (r_1, \ldots, r_m) \in \{0, \frac{1}{2}, 1\}^m$ which describes the voters' preferences between x and y. For each crisp profile, the collective preference will be obtained by means of a crisp aggregation operator.

Definition 1. *A* crisp aggregation operator (CAO) *is a mapping* $H : \{0, \frac{1}{2}, 1\}^m \longrightarrow \{0, \frac{1}{2}, 1\}$.

The interpretation of collective preference is consistent with the foregoing interpretation for individual preferences. So, $H(\boldsymbol{r})$ shows us if an alternative is collectively preferred to the other or the alternatives are collectively indifferent, according to whether $H(\boldsymbol{r})$ is 1 (x defeats y), 0 (y defeats x) or $\frac{1}{2}$ (x and y tie).

We now consider some properties of CAOs that are well known in the literature: *Symmetry*, *self-duality*, *monotonicity* and *unanimity*. Symmetry, also referred to as *anonymity* and *equality*, means that the collective preference depends only on the set of individual preferences, but not on which individuals have these preferences; i.e., all voters are treated equally. Self-duality, also referred to as *neutrality* (May [49]), says that if everyone reverses his/her preferences between both alternatives, then the collective preference is also reversed; i.e., the alternatives are treated equally. Monotonicity means that if the individual support for an alternative increases, then the outcome for this alternative can not be worse than in the first case. And unanimity says that the collective preference coincides with individual preferences when these are the same. A characterization of the CAOs which simultaneously satisfy the three first properties can be found in Fishburn [21, p. 56].

On the sequel we will use the following notation: Given $a \in \mathbb{R}$, we denote by $[a]$ the *integer part of* a, i.e., the largest integer smaller than or equal to a. Given $\boldsymbol{r}, \boldsymbol{s} \in \{0, \frac{1}{2}, 1\}^m$ and σ a permutation on $\{1, \dots, m\}$, we denote $\boldsymbol{r}_\sigma = (r_{\sigma(1)}, \dots, r_{\sigma(m)})$; $\mathbf{1} = (1, \dots, 1)$; $t\,\mathbf{1} = (t, \dots, t)$; $\boldsymbol{r} \geq \boldsymbol{s}$ will mean $r_i \geq s_i$ for all $i \in \{1, \dots, m\}$; and $\boldsymbol{r} > \boldsymbol{s}$ will denote $\boldsymbol{r} \geq \boldsymbol{s}$ and $\boldsymbol{r} \neq \boldsymbol{s}$.

Definition 2. *Let H be a CAO.*

1. *H is* symmetric *if for all crisp profile $\boldsymbol{r}$ and all permutation σ of $\{1, \dots, m\}$ it holds*
$$H(\boldsymbol{r}_\sigma) = H(\boldsymbol{r}).$$
2. *H is* self-dual *if for all crisp profile $\boldsymbol{r}$ it holds*
$$H(\mathbf{1} - \boldsymbol{r}) = 1 - H(\boldsymbol{r}).$$
3. *H is* monotonic *if for all pair of crisp profiles $\boldsymbol{r}$ and $\boldsymbol{s}$ it holds*
$$\boldsymbol{r} \geq \boldsymbol{s} \;\Rightarrow\; H(\boldsymbol{r}) \geq H(\boldsymbol{s}).$$
4. *H is* unanimous *if for all $t \in \{0, \frac{1}{2}, 1\}$ it holds*
$$H(t\,\mathbf{1}) = t.$$

Starting from the previous properties we can obtain some interesting consequences. We will use the following notation: the cardinal of a set will be denoted by $\#$; given a crisp profile $\boldsymbol{r}$, we denote

$$n_x(\boldsymbol{r}) = \#\{i \mid r_i = 1\}, \qquad n_y(\boldsymbol{r}) = \#\{i \mid r_i = 0\};$$

i.e., $n_x(\boldsymbol{r})$ is the number of individuals who prefer x to y, while $n_y(\boldsymbol{r})$ is the number of individuals who prefer y to x.

For each crisp profile $\boldsymbol{r}$, if H is a symmetric CAO, then $H(\boldsymbol{r})$ depends only on $n_x(\boldsymbol{r})$ and $n_y(\boldsymbol{r})$. If H is a self-dual CAO, then it is characterized by the set

$$H^{-1}(\{1\}) = \{\boldsymbol{r} \in \{0, \tfrac{1}{2}, 1\}^m \mid H(\boldsymbol{r}) = 1\},$$

since

$$H^{-1}(\{0\}) = \{\boldsymbol{r} \in \{0, \tfrac{1}{2}, 1\}^m \mid \mathbf{1} - \boldsymbol{r} \in H^{-1}(\{1\})\},$$
$$H^{-1}(\{\tfrac{1}{2}\}) = \{0, \tfrac{1}{2}, 1\}^m \setminus \left(H^{-1}(\{1\}) \cup H^{-1}(\{0\})\right).$$

Therefore, any self-dual CAO can be defined by means of the crisp profiles where the CAO takes the value 1. Based on this approach, we now show some of the most popular voting systems.

Definition 3.

1. *The* simple majority*, H_S, is the self-dual CAO defined by*

$$H(\boldsymbol{r}) = 1 \;\Leftrightarrow\; n_x(\boldsymbol{r}) > n_y(\boldsymbol{r}).$$

2. *The* absolute majority*, H_A, is the self-dual CAO defined by*

$$H(\boldsymbol{r}) = 1 \;\Leftrightarrow\; n_x(\boldsymbol{r}) > \frac{m}{2}.$$

3. *The* Pareto majority*, H_P, is the self-dual CAO defined by*

$$H(\boldsymbol{r}) = 1 \;\Leftrightarrow\; n_x(\boldsymbol{r}) > 0 \;\text{ and }\; n_y(\boldsymbol{r}) = 0.$$

4. *The* unanimous majority*, H_U, is the self-dual CAO defined by*

$$H(\boldsymbol{r}) = 1 \;\Leftrightarrow\; n_x(\boldsymbol{r}) = m.$$

5. *Given $\beta \in [\frac{1}{2}, 1)$, the* absolute special majority *Q_β is the self-dual CAO defined by*

$$H(\boldsymbol{r}) = 1 \;\Leftrightarrow\; n_x(\boldsymbol{r}) > \beta m.$$

Notice that all the previous voting systems have been defined by means of only $n_x(\boldsymbol{r})$, $n_y(\boldsymbol{r})$ and m. Then, their associated CAOs are symmetric.

Simple majority has been widely studied in the literature. It is worth emphasizing that the first axiomatic characterization of it was given by May [49]. Other characterizations of simple majority can be found in Fishburn [21, 22], Campbell [8, 9], Maskin [48], Campbell and Kelly [10, 11], Aşan and Sanver [1], Woeginger [68], Miroiu [51], Yi [82] and Llamazares [42].

In relation to absolute majority, it has been characterized by Fishburn [21, p. 60], while Pareto majority has been characterized by Sen [60, p. 76]. On the other hand, unanimous majority has been characterized by Woeginger [68] and Llamazares [42].

Clearly, absolute special majorities are located between absolute majority, for $\beta = 1/2$, and unanimous majority, for $\beta \geq (m-1)/m$. They have been studied by Fishburn [21, p. 67] (without self-duality assumption) and Ferejohn and Grether [18].

It is worth mentioning that the choice of a voting system is not trivial because all of them have several drawbacks. In order to explain this matter, we consider 1001 voters. The ordered pair $(n_x(\boldsymbol{r}), n_y(\boldsymbol{r}))$ represents the result of a ballot (obviously, $1001 - n_x(\boldsymbol{r}) - n_y(\boldsymbol{r})$ is the number of voters who are indifferent between x and y). Suppose that the result of a ballot is $(1, 0)$. In this case, under simple majority voting, x wins. So, in simple majority an alternative can be elected with very poor support. Moreover, in this situation —in fact, this happens when the result is $(n+1, n)$ or $(n, n+1)$, for some $n \geq 0$ such that $2n + 1 \leq m$— the winning alternative can change when a single turncoat alters his preference between x and y.

In order to avoid the problem of minimum support, we would be able to use absolute majority. In this case, the problem of minimum support disappears but we continue to have a problem of stability when the result of the ballot is $(501, 500)$ or $(500, 501)$. Furthermore, a new drawback appears because under absolute majority the winning alternative needs a hight quantity of votes. Consequently, there is a loss of decisiveness and, in many instances, there is no winning alternative. On the other hand, if we use absolute majority so that the winning alternative has a wide support, it is paradoxical that x wins if the result of the ballot is $(501, 500)$ but not when the result is $(500, 0)$.

Pareto majority shares with simple and absolute majorities similar drawbacks. Since x wins when the result of a ballot is $(1, 0)$, there are problems of minimum support and stability. Moreover, there exists a winning alternative only if the result of the ballot is $(n, 0)$ or $(0, n)$, with $n \geq 1$; therefore, there is a problem of decisiveness. For its part, unanimous majority has a large problem of decisiveness: It is the less decisive among self-dual, monotonic and no constant CAOs.

In relation to absolute special majorities, they have a problem of decisiveness because the winning alternative needs at least $[1001\,\beta]+1$ votes even the other alternative lacks support. Moreover, this loss of decisiveness increases as β increases.

The previous analysis shows that decisiveness and stability are conflicting concepts. Therefore, it seems very interesting to look for a balance between them. In order to achieve this objective, it is necessary to formalize these concepts.

The notion of decisiveness was used by Ferejohn and Grether [18], under the *strongness* name, for analyzing absolute special majorities.

Definition 4. *Given two CAOs H_1 and H_2, we say that H_1 is* as decisive as H_2 *if for all $\boldsymbol{r} \in \{0, \frac{1}{2}, 1\}^m$ it holds*

$$H_2(\boldsymbol{r}) = 1 \;\Rightarrow\; H_1(\boldsymbol{r}) = 1, \qquad\qquad H_2(\boldsymbol{r}) = 0 \;\Rightarrow\; H_1(\boldsymbol{r}) = 0.$$

Obviously, if H_1 is as decisive as H_2, then $H_1 = H_2$ or there exists some profile $\boldsymbol{r} \in \{0, \frac{1}{2}, 1\}^m$ such that $H_2(\boldsymbol{r}) = \frac{1}{2}$ and $H_1(\boldsymbol{r}) \neq \frac{1}{2}$, i.e., H_1 is *more decisive* than H_2.

When we come to define the notion of stability, it is necessary to bear in mind that for any no constant CAO there exist crisp profiles where an alternative stops winning when a single voter changes his/her preference. Consequently, we consider that a CAO is stable of grade q (q-stable) when given any profile where there exists a winning alternative, q voters can change their preferences without the other alternative becomes the winner. Since the case where an alternative can never win lacks interest, we also ask in the definition of q-stability that there be profiles where the change in the opinion of $q + 1$ individuals produces the switch of the winning alternative.

Definition 5. *Given $q \in \{0, 1, \ldots, m-1\}$, a CAO H is* q-stable *if it satisfies the following conditions:*

1. *For all* $\boldsymbol{r}, \boldsymbol{s} \in \{0, \frac{1}{2}, 1\}^m$ *such that* $\#\{i \mid r_i \neq s_i\} \leq q$,

$$H(\boldsymbol{r}) = 1 \;\Rightarrow\; H(\boldsymbol{s}) \geq \tfrac{1}{2}, \qquad\qquad H(\boldsymbol{r}) = -1 \;\Rightarrow\; H(\boldsymbol{s}) \leq \tfrac{1}{2}.$$

2. *There exist* $\boldsymbol{r}, \boldsymbol{s} \in \{0, \frac{1}{2}, 1\}^m$ *such that* $\#\{i \mid r_i \neq s_i\} = q + 1$ *satisfying* $H(\boldsymbol{r}) = 1$ *and* $H(\boldsymbol{s}) = 0$.

We note that Theorem 7 in the Subsection 4.2 shows what is the best voting system regarding a balance between decisiveness and stability.

3 Aggregation operators

In order that voters can show different levels of intensity between the alternatives, we consider $r_i \in [0, 1]$ instead of $r_i \in \{0, \frac{1}{2}, 1\}$. In this way, r_i denotes the *intensity* with which voter i prefers x to y. Under this assumption, it is usual to suppose that the preferences are *reciprocal*, i.e., $1 - r_i$ is the intensity with which individual i prefers y to x (on this see Bezdek *et al.* [4], Nurmi [57], Tanino [63], Nakamura [55], Świtalski [61, 62], García-Lapresta and Llamazares [29] and De Baets *et al.* [13], among others).

Similarly to the crisp case, a *profile* is a vector $\boldsymbol{r} = (r_1, \ldots, r_m) \in [0, 1]^m$ which describes the voters' preferences between x and y. For each profile, the collective preference will be obtained by means of an aggregation operator. These functions have been widely studied in the literature (see e.g. Dubois and Prade [17], Mizumoto [52, 53], Dubois and Koning [16], Yager [74], Fodor and Roubens [25], Grabisch *et al.* [34], Marichal [46], Calvo *et al.* [6], Xu and Da [72], Beliakov and Calvo [3], Mesiar *et al.* [50] and Torra [66], among other contributions).

Definition 6. *An* aggregation operator *is a mapping* $F : [0, 1]^m \longrightarrow [0, 1]$.

The properties introduced in Definition 2 for CAO's are also valid for aggregation operators, by considering general profiles instead of crisp profiles, and they have a similar interpretation. Moreover, we present an additional property for aggregation operators: *strict monotonicity*. This property means that the collective intensity of preference increases if no individual intensity decreases and some individual intensity increases.

Definition 7. *An aggregation operator* F *is* strictly monotonic *if for every pair of different profiles* $\boldsymbol{r}, \boldsymbol{s} \in [0, 1]^m$ *it holds*

$$\boldsymbol{r} > \boldsymbol{s} \;\Rightarrow\; F(\boldsymbol{r}) > F(\boldsymbol{s}).$$

Although there exist numerous classes of aggregation operators, on the sequel we only focus our study on OWA operators. Nevertheless, other aggregation operators such as quasiarithmetic means have been also considered in this context by García-Lapresta and Llamazares [29, 30] and Llamazares and García-Lapresta [44].

On the other hand, since discrete Sugeno and Choquet integrals have been widely used as aggregation operators in multi-criteria decision making problems, we next do some considerations. Kandel and Byatt [39] have proven that the discrete Sugeno integral can be expressed in terms of the median. Moreover, it satisfies a similar property to stability under positive linear transformations, but for ordinal values. For these reasons, as Grabisch [33] points out, the discrete Sugeno integral seems to be more suitable for ordinal aggregation.

In relation to discrete Choquet integral, it is well-known that these aggregation functions generalize OWA operators (see e.g. Murofushi and Sugeno [54] and Fodor *et al.* [24]). Furthermore Grabisch [32] has proven that the class of OWA operators coincides with the class of symmetric discrete Choquet integrals. Therefore, since symmetry is an essential property in Social Choice Theory, it is sufficient to consider OWA operators.

3.1 OWA operators

Yager [73] introduced the OWA operators as a tool for aggregation procedures in multi-criteria decision making. An OWA operator is similar to a weighted mean, but with the values of the variables previously ordered in a decreasing way. Thus, contrary to the weighted means, the weights are not associated with concrete variables and, therefore, they are symmetric. Moreover, they verify other interesting properties, such as monotonicity, unanimity, continuity and compensativeness, i.e., the value of an OWA operator is always located between the minimum and the maximum values of the variables. On the other hand, OWA operators generalize the maximum and the minimum operators, the arithmetic mean, the median and the k-order statistic. For these reasons, OWA operators have been widely used in the literature (see, for instance, Yager and Kacprzyk [81] and Calvo *et al.* [7]).

OWA operators have been characterized by Fodor *et al.* [24], Ovchinnikov [59] and Marichal and Mathonet [47]. On the other hand, there exist numerous generalizations of OWA operators: for instance, the quasi-OWA operators (Fodor *et al.* [24]), the Weighted Ordered Weighted Averaging (WOWA) Operators (Torra [64]), the Weighted Order Statistic Averaging (WOSA) operators (Ovchinnikov [59]), the Induced Ordered Weighted Averaging (IOWA) operators (Yager and Filev [80]), and the Heavy Ordered Weighted Averaging (HOWA) operators (Yager [78]).

OWA operators are usually defined as functions whose domain is $\mathbb{R}^m$. However, since individual intensities of preference vary between 0 and 1, we have restricted their domain to $[0,1]^m$.

Definition 8. *Let* $\boldsymbol{w} = (w_1, \ldots, w_m) \in [0,1]^m$ *satisfying* $\sum_{i=1}^{m} w_i = 1$*. The* OWA operator associated with $\boldsymbol{w}$ *is the aggregation operator* $F^{\boldsymbol{w}}$ *defined by*

$$F^{\boldsymbol{w}}(\boldsymbol{r}) = \sum_{i=1}^{m} w_i \, r_{\sigma(i)},$$

where σ is a permutation of $\{1,\ldots,m\}$ such that $r_{\sigma(1)} \geq \cdots \geq r_{\sigma(m)}$.

OWA operators are symmetric, monotonic and unanimous aggregation operators. Self-dual OWA operators have been characterized by Marichal [46, p. 103] and García-Lapresta and Llamazares [30], while characterizations of strictly monotonic OWA operators can be found in Marichal [46, p. 103] and Llamazares [43].

Proposition 1. *If $F^{\boldsymbol{w}}$ is an OWA operator, then:*

1. *$F^{\boldsymbol{w}}$ is self-dual if and only if $w_{m+1-i} = w_i$ for every $i \in \{1,\ldots,[\frac{m}{2}]\}$.*
2. *$F^{\boldsymbol{w}}$ is strictly monotonic if and only if $w_i > 0$ for every $i \in \{1,\ldots,m\}$.*

We will denote by $\mathcal{W}$ the set of weighting vectors associated with self-dual OWA operators, i.e.,

$$\mathcal{W} = \left\{ \boldsymbol{w} \in [0,1]^m \mid \sum_{i=1}^{m} w_i = 1 \text{ and } w_{m+1-i} = w_i \text{ for all } i \in \{1,\ldots,[\tfrac{m}{2}]\} \right\}.$$

One of the most important issues in the field of OWA operators is the determination of the associated weights. In order to solve this problem, several methods have appeared in the literature: quantifier guided aggregation (Yager [73, 75, 77]), exponential smoothing (Filev and Yager [20]), learning approach (Yager and Filev [80], Torra [65]), genetic algorithms (Nettleton and Torra [56]), linear objective-programming model under partial weight information (Xu and Da [71]), parametric geometric approach (Liu and Chen [40]), normal distribution based method (Xu [69]) and argument-dependent approach (Xu [70]; see Beliakov and Calvo [2] as well). Another important class of these methods is based on the *orness measure*. This concept was introduced by Yager [73] for characterizing the degree with which the aggregation is like an *or* operation. The orness of an OWA operator ranges between 0 and 1. Moreover, it takes the value 0 if and only if the OWA operator is the *minimum*, and it only takes the value 1 when the OWA operator is the *maximum*.

Definition 9. *The* orness measure *of an OWA operator $F^{\boldsymbol{w}}$ is defined by*

$$orness\,(\boldsymbol{w}) = \frac{1}{m-1} \sum_{i=1}^{m} (m-i)\, w_i.$$

O'Hagan [58] was the first to suggest the use of the vector which maximizes the entropy of the OWA weights for a given level of orness. His approach is based on the solution of the following constrained optimization problem:

$$\begin{aligned} &\max -\sum_{i=1}^{m} w_i \ln(w_i), \\ &\text{s.t.} \quad \mathit{orness}\,(\boldsymbol{w}) = \alpha, \quad 0 \le \alpha \le 1, \\ &\qquad w_i \ge 0, \quad i = 1, \ldots, m, \\ &\qquad \sum_{i=1}^{m} w_i = 1. \end{aligned} \tag{1}$$

The analytic properties of these maximal entropy OWA operators were studied by Filev and Yager [19]. Fullér and Majlender [26] transferred problem (1) to a polynomial equation, which is solved for determining the optimal weighting vector. To avoid the resolution of a nonlinear optimization problem, Yager [79] introduces a simpler procedure which tries to keep the spirit of maximizing the entropy for a given level of orness. Similar approaches (through a fixed level of orness) have also been proposed and solved analytically by Fullér and Majlender [27], Wang and Parkan [67] and Majlender [45]. The first authors suggest to select the vector which minimizes the variance of the weighting vector in order to obtain the minimal variability OWA weights. Thus, if we denote by $\boldsymbol{W}_\alpha$ the set of constraints of problem (1), they propose and solve the following mathematical programming problem:

$$\begin{aligned} &\min \frac{1}{m} \sum_{i=1}^{m} w_i^2 - \frac{1}{m^2}, \\ &\text{s.t.} \quad \boldsymbol{w} \in \boldsymbol{W}_\alpha. \end{aligned} \tag{2}$$

Wang and Parkan [67] suggest to minimize the disparities between adjacent weights. For this, they bring up the following constrained optimization problem:

$$\begin{aligned} &\min \max_{i \in \{1,\ldots,m-1\}} |w_i - w_{i+1}|, \\ &\text{s.t.} \quad \boldsymbol{w} \in \boldsymbol{W}_\alpha. \end{aligned} \tag{3}$$

For his part, Majlender [45] determines a parametric class of OWA operators having maximal Rényi entropy OWA weights. Given $\theta \in \mathbb{R}$, the parametric model proposed and solved by this author is the following:

$$\begin{aligned} &\max \log_2 \left(\sum_{i=1}^{m} w_i^\theta \right)^{1/(1-\theta)}, \\ &\text{s.t.} \quad \boldsymbol{w} \in \boldsymbol{W}_\alpha. \end{aligned} \tag{4}$$

However, in these models, the aggregation procedure is only taken into account when the level of orness is previously fixed, and the weights are determined through properties that only concern them (for instance, to maximize the entropy, to minimize the variance, etc.). Moreover, in order to guarantee an egalitarian treatment between the alternatives, we consider self-dual OWA operators whose level of orness is 0.5. Therefore, in our case, the methodologies based on fixing a given level of orness lack sense because the arithmetic mean is always the solution.

Because of the previous reasons, we propose the choice of OWA operator weights to take into account the class of majority rule that we want to obtain when individuals do not grade their preferences between the alternatives. For this purpose, the procedure to follow is described in the following subsection.

3.2 Obtaining CAOs from aggregation operators

When individuals have crisp preferences, every aggregation operator F can be restricted to crisp profiles:

$$F|_{\{0,\frac{1}{2},1\}^m} : \{0, \tfrac{1}{2}, 1\}^m \longrightarrow [0,1].$$

If we wish to obtain a CAO from F, then it will be necessary to obtain collective intensities of preference within $\{0, \frac{1}{2}, 1\}$ instead of $[0,1]$. It is possible to get these values by means of a procedure based on the α-cuts of F.

Definition 10. *Let F be an aggregation operator and $\alpha \in [\frac{1}{2}, 1)$. The α–*CAO *associated with F is the CAO F_α defined by*

$$F_\alpha(\boldsymbol{r}) = \begin{cases} 1, & \text{if } F(\boldsymbol{r}) > \alpha, \\ \frac{1}{2}, & \text{if } 1-\alpha \le F(\boldsymbol{r}) \le \alpha, \\ 0, & \text{if } F(\boldsymbol{r}) < 1-\alpha. \end{cases}$$

Thus, when individuals have crisp preferences, we can generate different CAOs from an aggregation operator F, by considering appropriate values of the parameter $\alpha \in [\frac{1}{2}, 1)$. Moreover, these CAOs, F_α, are symmetric, self-dual, monotonic and unanimous whenever the original aggregation operator F satisfies these properties.

4 Generating voting systems from OWA operators

In Definition 10 we provide a procedure which generates CAOs from an aggregation operator by means of a parameter $\alpha \in [\frac{1}{2}, 1)$. Now, we will use this procedure for obtaining the voting systems appearing in Definition 3. In this way, the following subsection is devoted to characterizing the OWA operators that allow us to generalize the mentioned voting systems. However, although we can obtain a specific CAO by means of a wide variety of OWA operators, not all of them are suitable, as we show in Subsection 4.2. For this reason, we also propose a procedure to determine the best-suited OWA operators.

Since the considered CAOs are self-dual, we only take into account self-dual OWA operators in order to guarantee that the obtained α-CAOs be self-dual too. The results contained in this section can be found in Llamazares [41, 43].

4.1 Characterization results

In the following theorems we characterize the OWA operators for which we can generate simple, absolute, Pareto, unanimous and absolute special majorities.

Theorem 1. *Let $F^{\boldsymbol{w}}$ be a self-dual OWA operator and $\alpha \in [\frac{1}{2}, 1)$. Then the following statements are equivalent:*

1. $F^{\boldsymbol{w}}_{\alpha} = H_S$.
2. $F^{\boldsymbol{w}}$ *is strictly monotonic and* $\alpha < \dfrac{1 + \min\{w_1, \dots, w_m\}}{2}$.

Theorem 2. *Let $F^{\boldsymbol{w}}$ be a self-dual OWA operator and $\alpha \in [\frac{1}{2}, 1)$. Then the following statements are equivalent:*

1. $F^{\boldsymbol{w}}_{\alpha} = H_A$.
2. a) *If m is odd:* $w_{\frac{m+1}{2}} > \frac{1}{3}$ *and* $\dfrac{3 - w_{\frac{m+1}{2}}}{4} \leq \alpha < \dfrac{1 + w_{\frac{m+1}{2}}}{2}$.

   b) *If m is even:* $w_{\frac{m}{2}} > \frac{1}{4}$ *and* $\frac{3}{4} \leq \alpha < \frac{1}{2} + w_{\frac{m}{2}}$.

Theorem 3. *Let $F^{\boldsymbol{w}}$ be a self-dual OWA operator and $\alpha \in [\frac{1}{2}, 1)$. Then the following statements are equivalent:*

1. $F^{\boldsymbol{w}}_{\alpha} = H_P$.
2. $w_1 > \frac{1}{3}$ *and* $1 - w_1 \leq \alpha < \dfrac{1 + w_1}{2}$.

Theorem 4. *Let $F^{\boldsymbol{w}}$ be a self-dual OWA operator and $\alpha, \beta \in [\frac{1}{2}, 1)$, with $[\beta m] > \frac{m}{2}$. Then the following statements are equivalent:*

1. $F^{\boldsymbol{w}}_{\alpha} = Q_{\beta}$.
2. $w_{m-[\beta m]} > \displaystyle\sum_{i=1}^{m-[\beta m]-1} w_i$ *and* $1 - \frac{1}{2} \displaystyle\sum_{i=1}^{m-[\beta m]} w_i \leq \alpha < 1 - \displaystyle\sum_{i=1}^{m-[\beta m]-1} w_i$.

As a particular case of this result (for $[\beta\, m] = m - 1$), we can obtain a characterization of self-dual OWA operators which generalize unanimous majority.

Corollary 1. *Let $F^{\boldsymbol{w}}$ be a self-dual OWA operator and $\alpha \in [\frac{1}{2}, 1)$. Then the following statements are equivalent:*

1. $F^{\boldsymbol{w}}_{\alpha} = H_U$.
2. $w_1 > 0$ *and* $\alpha \geq 1 - \dfrac{w_1}{2}$.

4.2 Choosing the best-suited OWA operators

As we have seen in Theorems 1, 2, 3, 4 and Corollary 1, there exist a lot of self-dual OWA operators which generate the same CAO. However, such as we will see in the following example, not all of them are suitable for a specific voting system.

Example 1. Consider $m = 5$, $\alpha = 0.504$ and $\boldsymbol{w} = (0.01, 0.01, 0.96, 0.01, 0.01)$. By Theorem 1, $F_\alpha^{\boldsymbol{w}} = H_S$. However, these weights are close to that of the vector $\boldsymbol{w'} = (0, 0, 1, 0, 0)$, and in this case, by Theorem 2, $F_\alpha^{\boldsymbol{w'}} = H_A$ for every $\alpha \in [\frac{1}{2}, 1)$ (in fact, with a suitable value of α, we would be able to choose the weights of $\boldsymbol{w}$ as close as we want to that of $\boldsymbol{w'}$). Therefore, although $F_\alpha^{\boldsymbol{w}} = H_S$, the choice of $\boldsymbol{w}$ and α does not seem the best for representing simple majority.

In order to avoid the previous situation, we take into account that for each self-dual OWA operator $F^{\boldsymbol{w}}$ and for each symmetric, monotonic and self-dual CAO H, the set $\{\alpha \in [\frac{1}{2}, 1) \mid F_\alpha^{\boldsymbol{w}} = H\}$ is an interval with endpoints $\underline{\alpha}(\boldsymbol{w}, H)$ and $\overline{\alpha}(\boldsymbol{w}, H)$ (it can be empty). In this way, we propose to choose the self-dual OWA operators that maximize the measure of interval, i.e., the value $\overline{\alpha}(\boldsymbol{w}, H) - \underline{\alpha}(\boldsymbol{w}, H)$. In the following theorem we show the weighting vectors that satisfy this condition for simple, absolute, Pareto and absolute special majorities.

Theorem 5. *Let H be a symmetric, monotonic and self-dual CAO and $\boldsymbol{w}^*$ be the weighting vector solution of the problem*

$$\max_{\boldsymbol{w} \in \mathcal{W}} \overline{\alpha}(\boldsymbol{w}, H) - \underline{\alpha}(\boldsymbol{w}, H).$$

Then:

1. *If $H = H_S$, then $F^{\boldsymbol{w}^*}$ is the arithmetic mean and*

$$F_\alpha^{\boldsymbol{w}^*} = H_S \;\Leftrightarrow\; \frac{1}{2} \leq \alpha < \frac{m+1}{2m}.$$

2. *If $H = H_A$, then $F^{\boldsymbol{w}^*}$ is the median. Moreover:*
   - *a) If m is odd, then $F_\alpha^{\boldsymbol{w}^*} = H_A$ for every $\alpha \in [\frac{1}{2}, 1)$.*
   - *b) If m is even, then $F_\alpha^{\boldsymbol{w}^*} = H_A \;\Leftrightarrow\; \frac{3}{4} \leq \alpha < 1$.*
3. *If $H = Q_\beta$, with $[\beta m] > \frac{m}{2}$, then*

$$w_i^* = \begin{cases} \frac{1}{2}, & \text{if } i = m - [\beta m], [\beta m] + 1, \\ 0, & \text{otherwise,} \end{cases}$$

and $F_\alpha^{\boldsymbol{w}^*} = Q_\beta \;\Leftrightarrow\; \frac{3}{4} \leq \alpha < 1$.

4. If $H = H_P$, then

$$w_i^* = \begin{cases} \frac{1}{2}, & \text{if } i = 1, m, \\ 0, & \text{otherwise,} \end{cases}$$

and $F_\alpha^{\boldsymbol{w}^*} = H_P \;\Leftrightarrow\; \frac{1}{2} \le \alpha < \frac{3}{4}$.

The obtained OWA operators are the arithmetic mean, the median and the average of the j-th and the $(m+1-j)$-th order statistics. Because they are the most suitable for generating some of the most important classes of voting systems, it would be interesting to know the α–CAOs associated with these OWA operators.

Since the arithmetic mean is a specific case of quasiarithmetic means, we can consider the result given by García-Lapresta and Llamazares [30, Proposition 4] for *majorities based on difference of votes* in the framework of quasiarithmetic means.

Theorem 6. *If $F^{\boldsymbol{w}}$ is the arithmetic mean, $\alpha \in [\frac{1}{2}, 1)$ and $k = [m(2\alpha - 1)]$, then $F_\alpha^{\boldsymbol{w}}$ coincides with M_k, the self-dual CAO defined by*

$$M_k(\boldsymbol{r}) = 1 \;\Leftrightarrow\; n_x(\boldsymbol{r}) > n_y(\boldsymbol{r}) + k.$$

The resultant CAOs are based on difference of votes: an alternative wins when the difference between the number of votes obtained by this alternative and that obtained by the other is greater than the quantity $[m(2\alpha-1)]$. These voting systems were introduced in García-Lapresta and Llamazares [30] and they have been recently analyzed by Llamazares [42] within the Social Choice approach. As well as the good properties appearing in the previous mentioned papers, it is worth emphasizing that they are the best voting systems that we look for at the end of Section 2 in order to achieve a balance between decisiveness and stability.

Theorem 7. *Given $k \in \{0, 1, \ldots, m-1\}$ and $k' \in \{k, \ldots, m-1\}$, the M_k majority is the most decisive symmetric, self-dual, monotonic and k'-stable CAO.*

In relation to the median, by 2.a) of Theorem 5, we have that $F_\alpha^{\boldsymbol{w}} = H_A$ for all $\alpha \in [\frac{1}{2}, 1)$ when m is odd. When m is even, the median is the average of the $m/2$-th and the $(m/2+1)$-th order statistics. Therefore, we can obtain the α–CAOs associated with it as a particular case of the following result.

Theorem 8. *Let $j \in \{1, \ldots, [\frac{m}{2}]\}$ and $F^{\boldsymbol{w}}$ be the OWA operator defined by*

$$w_i = \begin{cases} \frac{1}{2}, & \text{if } i = j, m+1-j, \\ 0, & \text{otherwise.} \end{cases}$$

Then the following statements are satisfied:

1. *If* $\frac{1}{2} \leq \alpha < \frac{3}{4}$*, then* $F_\alpha^{\boldsymbol{w}}$ *coincides with the self-dual CAO* H *defined by*

$$H(\boldsymbol{r}) = 1 \;\Leftrightarrow\; n_x(\boldsymbol{r}) \geq j \;\; \textit{and} \;\; n_y(\boldsymbol{r}) < j.$$

2. *If* $\frac{3}{4} \leq \alpha < 1$*, then* $F_\alpha^{\boldsymbol{w}} = Q_\beta$ *for* $1 - \frac{j}{m} \leq \beta < 1 - \frac{j-1}{m}$.

It is worth emphasizing that when $j = 1$, i.e., when the OWA operator is the average of the maximum and the minimum, the obtained α–CAOs are Pareto and unanimous majorities:

$$F_\alpha^{\boldsymbol{w}} = \begin{cases} H_P, & \text{if } \frac{1}{2} \leq \alpha < \frac{3}{4}, \\ H_U, & \text{if } \frac{3}{4} \leq \alpha < 1. \end{cases}$$

5 Concluding remarks

In this chapter we have considered voting situations where individuals only compare two alternatives. Although individuals usually feel different modalities of preference when they compare the feasible alternatives, classic voting systems require that voters show crisp preferences. Therefore, individuals are forced to identify very different circumstances. Consequently, the outcomes provided by the classic voting procedures could be no faithful with individual opinions and they could lead to inappropriate decisions.

We assume that individuals can show their intensities of preference by means of numerical values within the unit interval. In order to aggregate these intensities, we have considered aggregation operators. More concretely, we have focused our attention on OWA operators and we have presented some characterizations that determine which weighting vectors of self-dual OWA operators and which α-cuts allow to generate the genuine considered voting systems (simple, absolute, Pareto, unanimous and absolute special majorities), when individual preferences are crisp. In this sense, by satisfying the obtained conditions we can extend the mentioned classic voting systems to the case of individuals show their intensities of preference through numerical values within the unit interval.

Among the variety of self-dual OWA operators generating a specific classic voting system, we have found those weighting vectors which can be considered as the best-suited for extending the aforementioned classic voting systems. It is worth emphasizing that the founded OWA operators allow us to obtain new voting systems, some of them satisfying very interesting properties. For instance, through the arithmetic mean we can obtain majorities based on difference of votes which achieve a balance between decisiveness and stability. Therefore, the extension of classical voting systems to the field of gradual preferences allows us to find new voting systems which are the solution of some posed problems within the classic voting theory.

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