Consistent models of transitivity for reciprocal preferences on a finite ordinal scale

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Abstract

In this paper we consider a decision maker who shows his/her preferences for different alternatives through a finite set of ordinal values. We analyze the problem of consistency taking into account some transitivity properties within this framework. These properties are based on the very general class of conjunctors on the set of ordinal values. Each reciprocal preference relation on a finite ordinal scale has both a crisp preference and a crisp indifference relation associated to it in a natural way. Taking this into account, we have started by analyzing the problem of propagating transitivity from the preference relation on a finite ordinal scale to the crisp preference and indifference relations. After that, we carried out the analysis in the opposite direction. We provide some necessary and sufficient conditions for that propagation, and therefore, we characterize the consistent class of conjunctors in each direction.

Key words: Ordinary preference structures; reciprocal preferences on a finite ordinal scale; conjunctor; transitivity.

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1 Introduction

Decision making is present in many situations in life. Decision making is a process in which we can distinguish four basic phases (see Simon [30], among many others): the recollection of information phase, the design phase, the choice phase and the review phase. Preference models are an essential part of the design phase and, therefore, of the decision chain. This justifies the large number of studies on preference structures in classical or crisp set theory (see Roubens and Vincke [29], Pirlot and Vincke [28] and García-Lapresta and Rodríguez-Palmero[18], among many others). In order to model the preference intensities in a more effective way, some authors have considered fuzzy preferences (see Nurmi [25], Tanino [34], Fodor and Roubens [10], De Baets and Fodor [2] and García-Lapresta and Llamazares [12], among others). However, preferences on a finite ordinal scale could be more appropriate in order to capture the lack of precision in human behavior (see Zadeh [36,37], Grabisch [19], Herrera, Herrera-Viedma and Verdegay [20,21], García-Lapresta [11] and García-Lapresta, Martínez-Panero and Meneses [13], among others). We are now working in this direction and, in particular, we take into account the approach included in García-Lapresta [11].

Any preference on a finite ordinal scale has an ordinary preference structure associated to it. Therefore, we are interested in linking these two ways of measuring the preferences of a decision maker. We focus our attention on transitivity being an essential point to model rationality, so we relate the transitivity of a reciprocal preference on a finite ordinal scale with the transitivity of the associated crisp preference and indifference relations.

It is important to note that a wide class of linguistic models have been developed in order to deal with different problems within the decision making framework and related areas (see, for instance, [20,21,24,27,35,37–39]).

The structure of this paper is as follows: Section 2 will be devoted to review the notions of ordinary preference structure and reciprocal preference on a finite ordinal scale. In Section 3 we will introduce the definition of transitivity for reciprocal preference relations on a finite ordinal scale. We will propose two ways to generalize the classical concept of transitivity, strong and weak transitivity, so we will devote Section 4 to the strong case and Section 5 to the weak case. We will characterize the family of conjunctors with an appropriate behavior with respect to the propagation of the transitivity in each direction (ordinal reciprocal to crisp and conversely) in both sections. We will make a brief conclusion in the last section.

2 Basic concepts

Along the paper, we will handle basic concepts and properties concerning classical and ordinal reciprocal preference relations. In this section, we briefly review them.

Let S, T be two ordinary binary relations on A. S^{-1} is the *inverse relation* of S $[a\,S^{-1}\,b \;\Leftrightarrow\; b\,S\,a]$; S^c is the *complement relation* of S $[a\,S^c\,b \;\Leftrightarrow\; \neg(a\,S\,b)]$; $S \cap T$ is the *intersection relation* of S and T $[a\,(S \cap T)\,b \;\Leftrightarrow\; (a\,S\,b \;\wedge\; a\,T\,b)]$; $S \cup T$ is the *union relation* of S and T $[a\,(S \cup T)\,b \;\Leftrightarrow\; (a\,S\,b \;\vee\; a\,T\,b)]$; $S \circ T$ is the *composition relation* of S and T $[a\,(S \circ T)\,b \;\Leftrightarrow\; \exists\, c \in A \;\; (a\,S\,c \;\wedge\; c\,T\,b)]$.

2.1 Classical preference structures

A way of introducing classical preference and indifference concepts is taking the *(strong) preference* as primitive notion, through an ordinary binary relation P on a set of alternatives A, where $a\,P\,b$ means "a is preferred to b" or "a is better than b". A basic assumption for P is asymmetry: $P \cap P^{-1} = \emptyset$. In this case the *indifference relation*, I, is defined by absence of preference, i.e., a is indifferent to b when neither a is preferred to b nor b is preferred to a: $I = (P \cup P^{-1})^c = P^c \cap (P^{-1})^c$. Then, the *weak preference* (or *preference-indifference*) relation is defined by $P \cup I = (P^{-1})^c$.

The above model is equivalent to the notion of ordinary preference structure.

Definition 1 *An* ordinary preference structure *on a set of alternatives A is a couple (P, I) of ordinary binary relations on A satisfying:*

(1) P is asymmetric.
(2) I is symmetric.
(3) $P \cap I = \emptyset$.
(4) $P \cup P^{-1} \cup I = A \times A$.

P, I and $P \cup I$ are called *preference*, *indifference* and *weak preference* (or *preference-indifference*) relations, respectively.

We note that Roubens and Vincke [29, 2.2], among others, consider another ordinary binary relation in the definition of preference structure, *incomparability.* However, in this paper we consider indifference as absence of preference. In this sense, incomparability is a specific case of indifference. Then, for each pair of alternatives $a, b \in A$, one and only one of the following situations happens: $a\,P\,b$ or $b\,P\,a$ or $a\,I\,b$.

2.2 Ordinal reciprocal preference relations

Let A be a finite set of alternatives with cardinal[1] greater than 2 and let $\mathcal{L}_n = \{l_0, l_1, \dots, l_{2n}\}$ be a set of linguistic labels ranked by a linear order: $l_0 < l_1 < \cdots < l_{2n}$, where $n \geq 1$. The intermediate label l_n represents indifference, and the rest of labels are defined around it symmetrically. In the following sections, two subsets of $\mathcal{L}_n$ will play an important role, which are $\mathcal{L}_n^{+} = \{l_n, \dots, l_{2n}\}$ and $\mathcal{L}_n^{++} = \{l_{n+1}, \dots, l_{2n}\}$. They will correspond to the degrees of weak preference and strict preference, respectively.

We now introduce reciprocal preference relations on a finite ordinal scale, similarly to García-Lapresta, Martínez-Panero and Meneses [13]. This definition is based on the proposed ideas for reciprocity in fuzzy binary relations (see Nurmi [25], Tanino [34] and García-Lapresta and Llamazares [12], among others). In the framework of fuzzy preference relations, reciprocity means that $R(a,b) + R(b,a) = 1$, for all $a, b \in A$. This property is equivalent to the following: $R(a,b) = k$ and $R(b,a) = k'$ imply that $k + k' = 1 = \max[0,1]$, for all $a, b \in X$ and $k, k' \in [0,1]$. Taking into account this idea, we can extend the notion of reciprocity to the framework of linguistic preferences in the following sense: $R(a,b) = l_k$ and $R(b,a) = l_{k'}$ imply that $k + k' = 2n = \max\{0, 1, \dots, 2n\}$, for all $a, b \in A$ and $k, k' \in \{0, 1, \dots, 2n\}$. But this is equivalent to $R(a,b) = l_k$ implies $R(b,a) = l_{2n-k}$, for all $a, b \in A$ and $k \in \{0, 1, \dots, 2n\}$.

Definition 2 *A* reciprocal preference relation on A based on $\mathcal{L}_n$ *is a mapping*

$$R : A \times A \longrightarrow \mathcal{L}_n$$

satisfying the reciprocity condition, that is,

$$R(a,b) = l_k \quad \Rightarrow \quad R(b,a) = l_{2n-k},$$

for all $a, b \in A$ and $k \in \{0, 1, \dots, 2n\}$.

We denote by $\mathcal{L}_n(A)$ the set of the reciprocal preference relations on A based on $\mathcal{L}_n$. Notice that $R \in \mathcal{L}_n(A)$ implies $R(a,a) = l_n$, for every $a \in A$.

The meaning of the labels can be summarized in this way:

[1] As we will work with transitivity, this requirement is necessary for considering transitivity in a proper and logical way.

(1) $R(a,b) = l_{2n}$, if a is totally preferred to b,
(2) $l_n < R(a,b) < l_{2n}$, if a is somewhat preferred to b,
(3) $R(a,b) = l_n$, if a is indifferent to b,
(4) $l_0 < R(a,b) < l_n$, if b is somewhat preferred to a,
(5) $R(a,b) = l_0$, if b is totally preferred to a.

Since l_k and l_{2n-k} represent the same modality of preference, used in a symmetric way, in $\mathcal{L}_n$ there are $n+1$ possibilities for declaring opinions between pairs of alternatives: n kinds for preference and 1 for indifference.

We note that for $\mathcal{L}_1 = \{l_0, l_1, l_2\}$, the only relation $R \in \mathcal{L}_1(A)$, which is defined by

$$R(a,b) = \begin{cases} l_0, \text{ if } b \text{ is preferred to } a, \\ l_1, \text{ if } a \text{ is indifferent to } b, \\ l_2, \text{ if } a \text{ is preferred to } b, \end{cases}$$

consists on the classical preference model.

When other modalities of preference are allowed, we should use $\mathcal{L}_n$ for $n > 1$. In fact, the sets of linguistic labels $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_4$ have been widely used in the literature.

Moreover, it is possible to associate an ordinary preference structure to any reciprocal preference relation. Thus, let $R \in \mathcal{L}_n(A)$ and let P_R be the ordinary binary relation on A defined by $a\, P_R\, b$ if $R(a,b) > l_n$. It is easy to see that P_R is asymmetric, hence an ordinary preference relation. We say that P_R is the *classical strict preference relation associated with* R. The *classical indifference relation associated with* R is just the indifference relation associated with P_R: $a\, I_R\, b$ if $R(a,b) = l_n$. We also say that (P_R, I_R) is the *ordinary preference structure associated with* R.

Example 3 *In order to consider an example based on a real case, we will consider the data included in García-Lapresta and Meneses [15]. In this paper, a group of 85 students were asked about their preferences over the following degrees: (a) Business Administration and Management, (b) Business Administration, (c) Law, (d) Business Administration, Management and Law, (e) Labor Relations and (f) Economics. Students had to compare each pair of alternatives through four modalities of preference: "totally", "highly", "rather" and "slightly", when they preferred one alternative to another; in absence of preference between alternatives they could declare "indifference". Thus, the set of linguistic labels is* $\mathcal{L}_8$ *where* l_4*=indifference,* l_5*=slightly preferred,* l_6*=rather preferred,* l_7*=highly preferred and* l_8*=totally preferred.*

Let us suppose the collective intensities of preference between all the pairs of alternatives is

R	a	b	c	d	e	f
a	l_4	l_5	l_6	l_5	l_7	l_5
b	l_3	l_4	l_5	l_3	l_5	l_3
c	l_2	l_3	l_4	l_2	l_5	l_2
d	l_3	l_5	l_6	l_4	l_6	l_4
e	l_1	l_3	l_3	l_2	l_4	l_2
f	l_3	l_5	l_6	l_4	l_6	l_4

then $R \in \mathcal{L}_n(A)$ *and the ordinary preference and indifference relations associated to* R *are*

P_R	a	b	c	d	e	f
a	0	1	1	1	1	1
b	0	0	1	0	1	0
c	0	0	0	0	1	0
d	0	1	1	0	1	0
e	0	0	0	0	0	0
f	0	1	1	0	1	0

I_R	a	b	c	d	e	f
a	1	0	0	0	0	0
b	0	1	0	0	0	0
c	0	0	1	0	0	0
d	0	0	0	1	0	1
e	0	0	0	0	1	0
f	0	0	0	1	0	1

Remark 4 Although every reciprocal preference relation has associated a preference structure, it is evident that this preference structure could come from different reciprocal preference relations. For instance, given a preference structure (P, I) on A, we can define the family of relations R_k, $k \in \{0, 1, \ldots, n-1\}$, as follows

$$R_k(a,b) = \begin{cases} l_{2n-k}, & \text{if } \; a\,P\,b, \\ l_n, & \text{if } \; a\,I\,b, \\ l_k, & \text{if } \; b\,P\,a. \end{cases}$$

It is immediate that $R_k \in \mathcal{L}_n(A)$ and (P, I) is the ordinary preference structure associated with R_k, for any $k \in \{0, 1, \ldots, n-1\}$.

3 Transitivity

In classical preferences, transitivity is essential for modelling rationality. The same happens for fuzzy preferences, but in this case, there is a wide class of transitivity notions for extending this property to the fuzzy framework (see Dubois and Prade [8], Ovchinnikov [26], Jain [22], Dasgupta and Deb [1], Switalski [31–33], García-Lapresta and Meneses [14–16], and García-Lapresta and Montero [17], among many others). Moreover, some interesting extensions of transitivity within the probabilistic choice framework can be found in Fishburn [9].

In this paper, we consider some models of rational behavior within the linguistic approach. In order to define the concept of transitivity for a reciprocal preference relation, since any relation $R \in \mathcal{L}_n(A)$ assumes values in the finite set $\mathcal{L}_n$, we could consider the idea of t-norm on a finite set (see Mayor and Torrens [23]), as it is usually generalized in the fuzzy context. However, for this purpose, associativity, commutativity and general boundary conditions are not required, as it is the case for additive fuzzy preference structures (see Díaz, De Baets and Montes [3,4] and Díaz, Montes and De Baets [6,7]). Without these properties, we are going to work only with the monotonicity condition and a specific boundary condition. This guarantees that the operator preserves the original notion of transitivity, if R takes only crisp degrees of preference. This boundary condition is based on the following results.

Proposition 5 *Let F be a monotonic operator $F : \mathcal{L}_n^+ \times \mathcal{L}_n^+ \to \mathcal{L}_n$. For any $R \in \mathcal{L}_n(A)$ such that*

$$R(a,b) \in \{l_0, l_n, l_{2n}\}, \forall a, b \in A,$$

$$R(a,c) \geq F(R(a,b), R(b,c)), \forall a, b, c \in A \text{ with } R(a,b), R(b,c) \in \mathcal{L}_n^+,$$

the associated weak preference $P_R \cup I_R$ is transitive if and only

$$\max\{F(l_n, l_{2n}), F(l_{2n}, l_n)\} > l_n \text{ or } F(l_n, l_n) > l_0.$$

PROOF. Let us suppose that $\max\{F(l_n, l_{2n}), F(l_{2n}, l_n)\} \leq l_n$ and $F(l_n, l_n) = l_0$. The relation R, defined by $R(b,c) = l_{2n}$, $R(c,b) = l_0$ and the value of R is l_n in any other case, belongs to $\mathcal{L}_n(A)$. Moreover, R fulfills that $R(x,y) \in \{l_0, l_n, l_{2n}\}, \forall x, y \in A$ and $R(x,z) \geq F(R(x,y), R(y,z)), \forall x, y, z \in A$ with $R(x,y), R(y,z) \in \mathcal{L}_n^+$. However, $P_R \cup I_R$ is not transitive, since $c(P_R \cup I_R)a$, $a(P_R \cup I_R)b$, but $b(P_R \cup I_R)c$.

Conversely, let us consider a monotonic operator F fulfilling $F(l_n, l_n) > l_0$ or $\max\{F(l_n, l_{2n}), F(l_{2n}, l_n)\} > l_n$ and a relation $R \in \mathcal{L}_n(A)$ such that

$R(a,b) \in \{l_0, l_n, l_{2n}\}, \forall a,b \in A$ and $R(a,c) \geq F(R(a,b), R(b,c)), \forall a,b,c \in A$ with $R(a,b), R(b,c) \in \mathcal{L}_n^+$. For any $a,b,c \in A$ such that $a(P_R \cup I_R)b$, $b(P_R \cup I_R)c$, then $R(a,b), R(b,c) \in \{l_n, l_{2n}\}$. If $F(l_n, l_n) > l_0$, then $R(a,c) \geq F(R(a,b), R(b,c)) \geq F(l_n, l_n) > l_0$ and therefore $R(a,c) \geq l_n$, that is, $a(P_R \cup I_R)c$. Otherwise, if $\max\{F(l_n, l_{2n}), F(l_{2n}, l_n)\} > l_n$, let us suppose $R(a,c) < l_n$, which is equivalent to $R(c,a) = l_{2n}$. Then we have two cases:

(1) If $F(l_n, l_{2n}) > l_n$, then

$$R(b,a) \geq F(R(b,c), R(c,a)) \geq F(l_n, l_{2n}) > l_n$$

which is a contradiction with $R(a,b) \geq l_n$.

(2) Analogously, if $F(l_{2n}, l_n) > l_n$, then

$$R(c,b) \geq F(R(c,a), R(a,b)) \geq F(l_{2n}, l_n) > l_n$$

which is a contradiction with $R(b,c) \geq l_n$.

Thus, in any case, $R(a,c) \geq l_n$, that is, $a(P_R \cup I_R)c$ and therefore $P_R \cup I_R$ is transitive. □

As it will be explained before Definition 8, in some cases it could be natural to require the "transitivity" only for strict preference intensities. In that cases, we could only assure the transitivity of the associated strict preference relation. In next proposition, the characterization of the monotonic operators with a consistent behavior in this sense is presented.

Proposition 6 *Let F be a monotonic operator $F : \mathcal{L}_n^+ \times \mathcal{L}_n^+ \to \mathcal{L}_n$. For any $R \in \mathcal{L}_n(A)$ such that*

$$R(a,b) \in \{l_0, l_n, l_{2n}\}, \forall a,b \in A,$$
$$R(a,c) \geq F(R(a,b), R(b,c)), \forall a,b,c \in A \text{ with } R(a,b), R(b,c) \in \mathcal{L}_n^{++},$$

the associated ordinary preference P_R is transitive if and only

$$F(l_{2n}, l_{2n}) > l_n.$$

PROOF. Let us suppose that $F(l_{2n}, l_{2n}) \leq l_n$, then the relation R, defined by $R(a,b) = R(b,c) = l_{2n}$, $R(b,a) = R(c,b) = l_0$ and R takes the value l_n in any other case, belongs to $\mathcal{L}_n(A)$. Moreover, R fulfills that $R(x,y) \in \{l_0, l_n, l_{2n}\}, \forall x,y \in A$ and $R(x,z) \geq F(R(x,y), R(y,z)), \forall x,y,z \in A$ satisfying that $R(x,y), R(y,z) \in \mathcal{L}_n^{++}$. However, P_R is not transitive, since aP_Rb, bP_Rc, but aI_Rc.

Conversely, let us consider a monotonic operator F fulfilling $F(l_{2n}, l_{2n}) > l_n$ and a relation $R \in \mathcal{L}_n(A)$ such that $R(a,b) \in \{l_0, l_n, l_{2n}\}, \forall a,b \in A$ and $R(a,c) \geq F(R(a,b), R(b,c)), \forall a,b,c \in A$ with $R(a,b), R(b,c) \in \mathcal{L}_n^{++}$. For any $a,b,c \in A$ such that aP_Rb and bP_Rc, then $R(a,b) = R(b,c) = l_{2n}$. This implies that $R(a,c) \geq F(l_{2n}, l_{2n}) > l_n$ and therefore $R(a,c) = l_{2n}$, that is, aP_Rc. Thus, we have proven that P_R is transitive. □

From Propositions 5 and 6, we obtain the natural requirements an operator must satisfy in order to define the transitivity as a generalization of the classical concept. Thus, we will consider a wider class to define transitivity than the class of t-norms: the class of conjunctors on $\mathcal{L}_n^+$.

Definition 7 *A* conjunctor *on $\mathcal{L}_n^+$ is a mapping $F : \mathcal{L}_n^+ \times \mathcal{L}_n^+ \longrightarrow \mathcal{L}_n$ satisfying the following condition:*

- *Monotonicity: $F(l_j, l_i) \leq F(l_k, l_i)$ and $F(l_i, l_j) \leq F(l_i, l_k)$, for all $l_i, l_j, l_k \in \mathcal{L}_n^+$ such that $l_j \leq l_k$.*
- *Boundary:* *(B1)* $\max\{F(l_n, l_{2n}), F(l_{2n}, l_n)\} > l_n$

  or (B2) $F(l_n, l_n) > l_0$ *and* $F(l_{2n}, l_{2n}) > l_n$.

The concept of monotonic operator (conjunctor without the boundary condition) was already considered in Díaz, García-Lapresta and Montes [5]. In that preliminary work, we presented some first results on the propagation of the transitivity based on monotonic operators. This paper largely extends that contribution for conjunctors.

If we consider the partial order relation on the class of conjunctors on $\mathcal{L}_n^+$ defined by

$$F \leq G \Leftrightarrow \forall l_i, l_j \in \mathcal{L}_n^+, \text{ if } F(l_i, l_j) = l_p \text{ and } G(l_i, l_j) = l_q, \text{ then } p \leq q,$$

the greatest conjunctor, denoted by F_G, is

$$F_G(l_i, l_j) = l_{2n}, \forall l_i, l_j \in \mathcal{L}_n^+,$$

and the minimal elements in the class of conjunctors on $\mathcal{L}_n^+$ are

$$F_{L1}(l_i, l_j) = \begin{cases} l_{n+1} & \text{if } \min\{l_i, l_j\} = l_{2n}, \\ l_1 & \text{otherwise,} \end{cases}$$

$$F_{L2}(l_i, l_j) = \begin{cases} l_{n+1} & \text{if } l_j = l_{2n}, \\ l_0 & \text{otherwise,} \end{cases} \qquad F_{L3}(l_i, l_j) = \begin{cases} l_{n+1} & \text{if } l_i = l_{2n}, \\ l_0 & \text{otherwise.} \end{cases}$$

There does not exist the lowest conjunctor.

Other examples of conjunctors are t-norms and t-conorms on $\mathcal{L}_n^+$ [23], in particular the mininum t-norm ($T_M(l_i, l_j) = \min\{l_i, l_j\}, \forall l_i, l_j \in \mathcal{L}_n^+$) and the maximum t-conorm ($S_M(l_i, l_j) = \max\{l_i, l_j\}, \forall l_i, l_j \in \mathcal{L}_n^+$).

From any conjunctor F, the usual way to define the transitivity of the reciprocal relation R is by means of the inequality $F(R(a,b), R(b,c)) \leq R(a,c)$, $\forall a, b, c \in A$ such that $R(a,b), R(b,c) \in \mathcal{L}_n^+$. However, this transitivity could be too strong when preferences try to capture the imprecision of human behavior. In fact, it has been shown in empirical analysis (see [14]) that the problem of irrationality in human answers is very common, mainly when indifference appears. Thus, an alternative idea is not to require transitivity for the indifference, but only for the strict preference relation. From these approaches, two different generalizations of the concept of transitivity appear.

Definition 8 *Let F be a conjunctor on $\mathcal{L}_n^+$ and let $R \in \mathcal{L}_n(A)$.*

(1) R *is* strongly F-transitive *if* $R(a,b) \geq F(R(a,c), R(c,b))$ *for all* $a, b, c \in A$, *whenever* $R(a,c), R(c,b) \in \mathcal{L}_n^+$.
(2) R *is* weakly F-transitive *if* $R(a,b) \geq F(R(a,c), R(c,b))$ *for all* $a, b, c \in A$, *whenever* $R(a,c), R(c,b) \in \mathcal{L}_n^{++}$.

In order to simplify the notation along this paper, we will consider the following subsets of $\mathcal{L}_n(A)$:

$$\mathcal{L}_n^{sF}(A) = \{R \in \mathcal{L}_n(A) \mid R \text{ is strongly } F\text{-transitive}\},$$
$$\mathcal{L}_n^{wF}(A) = \{R \in \mathcal{L}_n(A) \mid R \text{ is weakly } F\text{-transitive}\}.$$

Remark 9 • *It is trivial that for any conjunctor F, the strong F-transitivity of a relation $R \in \mathcal{L}_n(A)$ such that the degree of preference is crisp, that is, $R(a,b) \in \{l_0, l_n, l_{2n}\}, \forall a, b \in A$, implies the transitivity of $P_R \cup I_R$ and the weak F-transitivity of R implies the transitivity of P_R.*

- *Obviously, if R is strongly, respectively weakly, F-transitive and G is another conjunctor such that $G \leq F$, then R is also strongly, respectively weakly, G-transitive.*
- *Moreover, if R is strongly F-transitive, then it is clear that it is also weakly F-transitive, that is, $\mathcal{L}_n^{sF}(A) \subseteq \mathcal{L}_n^{wF}(A)$. However, the converse is not true, as the following example proves.*

  Let $A = \{a, b, c\}$ be the set of alternatives, let F be the conjunctor defined by

$$F_{LSI}(l_i, l_j) = \begin{cases} l_{n+1} & \text{if } \min\{l_i, l_j\} = l_{2n}, \\ l_n & \text{otherwise,} \end{cases}$$

and let r be a natural number in the interval $[0, n]$. Consider the reciprocal preference relation on A based on $\mathcal{L}_n$ given by

R	a	b	c
a	l_n	l_{n+r}	l_{n-r}
b	l_{n-r}	l_n	l_n
c	l_{n+r}	l_n	l_n

For $r = 0$, R is strongly, and therefore weakly, F_{LSI}-transitive.

For $r = 1$, R is weakly F_{LSI}-transitive, but it is not strongly F_{LSI}-transitive:

$$F_{LSI}(R(a,b), R(b,c)) = F_{LSI}(l_{n+1}, l_n) = l_n > l_{n-1} = R(a,c).$$

For $r = n$, R is neither strongly nor weakly F_{LSI}-transitive:

$$F_{LSI}(R(c,a), R(a,b)) = F_{LSI}(l_{2n}, l_{2n}) = l_{n+1} > l_n = R(c,b).$$

Example 10 *If we consider again the reciprocal preference relation R introduced in Example 3, R is strongly S_M-transitive and therefore, strongly T_M-transitive and weakly S_M-transitive. However, it is not weakly F_G-transitive, since*

$$F_G(R(a,b), R(b,c)) = F_G(l_5, l_5) = l_8 > l_6 = R(a,c),$$

and therefore, it is neither strongly F_G-transitive.

It is trivial that the relation that connects all the alternatives by the indifference label is weakly F-transitive for any conjunctor F, which implies $\mathcal{L}_n^{wF}(A) \neq \emptyset$. However, an extra requirement has to be imposed to the conjunctor F in order to assure $\mathcal{L}_n^{sF}(A) \neq \emptyset$, as it is proven in the following lemma.

Lemma 11 *Let F be a conjunctor on $\mathcal{L}_n^+$. The following statements hold:*

(1) $\mathcal{L}_n^{sF}(A) \neq \emptyset \Leftrightarrow F(l_n, l_n) \leq l_n$.
(2) $\mathcal{L}_n^{wF}(A) \neq \emptyset$.

PROOF.

(1) If $R \in \mathcal{L}_n^{sF}(A)$, then we have $l_n = R(a,a) \geq F(R(a,a), R(a,a))$ for every $a \in A$. Therefore, $F(l_n, l_n) \leq l_n$. Assume now that F satisfies $F(l_n, l_n) \leq l_n$. Then, the relation R defined as $R(a,b) = l_n$ for any $a, b \in A$ is strongly F-transitive.

(2) The relation R defined as $R(a,b) = l_n$ for any $a,b \in A$ is weakly F-transitive in a trivial manner. □

In Example 10, we proved that $\mathcal{L}_n^{sS_M} \neq \emptyset$. This is logical, since $S_M(l_n, l_n) = l_n$. However, from Lemma 11, $\mathcal{L}_n^{sF_G} = \emptyset$.

We have concluded the section devoted to properly define the concept of transitivity for ordinal reciprocal preference relations. In the following sections we are going to start by characterizing the class of conjunctors which has an appropriate behavior in the propagation from the F-transitivity of R to the transitivity of P_R and I_R, in the strong and weak cases. Then, given the close relationship between the transitivity of P_R and I_R and the properties $P_R \circ I_R \subseteq P_R$, $I_R \circ P_R \subseteq P_R$ (see Roubens and Vincke [29, 3.3] and García-Lapresta and Rodríguez-Palmero [18], among others), we will also study these two properties when P_R and I_R are obtained from a F-transitive R both in the strong and weak cases. Finally, the converse implication will be studied as well and the appropriate conjunctors characterized.

4 The strong case

In order to simplify the statement of the following results, let us define the class of conjunctors with an appropriate behavior with respect to the propagation of the strong transitivity.

Definition 12 *Let F be a conjunctor on $\mathcal{L}_n^+$. We say that F is a*

- SI conjunctor *if $\mathcal{L}_n^{sF}(A) \neq \emptyset$ and I_R is transitive for every $R \in \mathcal{L}_n^{sF}(A)$.*
- SP conjunctor *if $\mathcal{L}_n^{sF}(A) \neq \emptyset$ and P_R is transitive for every $R \in \mathcal{L}_n^{sF}(A)$.*
- SPI conjunctor *if $\mathcal{L}_n^{sF}(A) \neq \emptyset$ and $P_R \circ I_R \subseteq P_R$ for every $R \in \mathcal{L}_n^{sF}(A)$.*
- SIP conjunctor *if $\mathcal{L}_n^{sF}(A) \neq \emptyset$ and $I_R \circ P_R \subseteq P_R$ for every $R \in \mathcal{L}_n^{sF}(A)$.*

Next we characterize those conjunctors F that ensure that the indifference relation associated to a strongly F-transitive reciprocal relation is transitive.

Proposition 13 *Let F be a conjunctor on $\mathcal{L}_n^+$. F is a* SI *conjunctor if and only if one of the following conditions is satisfied:*

(1) $F(l_n, l_n) = l_n$.
(2) $F(l_n, l_n) < l_n < \max\{F(l_n, l_{n+1}), F(l_{n+1}, l_n)\}$.

PROOF. Let us suppose F is a SI conjunctor. Then, it follows from Definition 12, $\mathcal{L}_n^{sF}(A) \neq \emptyset$. Thus, by Lemma 11, $F(l_n, l_n) \leq l_n$. Let us suppose that

F satisfies $F(l_n, l_n) < l_n$ and

$\max\{F(l_n, l_{n+1}), F(l_{n+1}, l_n)\} \leq l_n$. Let $a, b, c \in A$ and $R \in \mathcal{L}_n(A)$ defined by $R(a, b) = R(b, a) = R(b, c) = R(c, b) = l_n$, $R(a, c) = l_{n-1}$, $R(c, a) = l_{n+1}$ and $R(d, e) = l_n$ for any $d, e \in A \setminus \{a, b, c\}$. It is easy to see that $R \in \mathcal{L}_n^{sF}(A)$. However, I_R is not transitive: $a\, I_R\, b$, $b\, I_R\, c$, but $c\, P_R\, a$. Consequently, F is not a SI conjunctor, which is a contradiction.

Conversely, assume F is a conjunctor on $\mathcal{L}_n^+$ satisfying $F(l_n, l_n) = l_n$ or $F(l_n, l_n) < l_n < \max\{F(l_n, l_{n+1}), F(l_{n+1}, l_n)\}$, then $F(l_n, l_n) \leq l_n$ and therefore, by Lemma 11, $\mathcal{L}_n^{sF}(A) \neq \emptyset$. Let now $R \in \mathcal{L}_n^{sF}(A)$ and $a, b, c \in A$ such that $a\, I_R\, b$ and $b\, I_R\, c$.

(1) If $F(l_n, l_n) = l_n$, then $R(a, c) \geq F(R(a, b), R(b, c)) = F(l_n, l_n) = l_n$ and, simultaneously, $R(c, a) \geq F(R(c, b), R(b, a)) = F(l_n, l_n) = l_n$. Consequently, we have $R(a, c) = R(c, a) = l_n$, i.e., $a\, I_R\, c$.
(2) If $F(l_n, l_{n+1}) > l_n$, suppose that $R(a, c) \neq l_n$.
  (a) If $R(a, c) > l_n$, then $l_n = R(b, c) \geq F(R(b, a), R(a, c)) \geq F(l_n, l_{n+1})$, which is a contradiction.
  (b) If $R(a, c) < l_n$, then $l_n = R(b, a) \geq F(R(b, c), R(c, a)) \geq F(l_n, l_{n+1})$, which is a contradiction.
(3) If $F(l_{n+1}, l_n) > l_n$, suppose that $R(a, c) \neq l_n$.
  (a) If $R(a, c) > l_n$, then $l_n = R(a, b) \geq F(R(a, c), R(c, b)) \geq F(l_{n+1}, l_n)$, which is a contradiction.
  (b) If $R(a, c) < l_n$, then $l_n = R(c, b) \geq F(R(c, a), R(a, b)) \geq F(l_{n+1}, l_n)$, which is a contradiction.

Consequently, if F satisfies the required conditions, then I_R is transitive. Hence, F is a SI conjunctor. □

An example of a *SI* conjunctor is any t-norm T or t-conorm S on $\mathcal{L}_n^+$, since $T(l_n, l_n) = S(l_n, l_n) = l_n$. However, the class of *SI* conjunctors is a proper subset of the class of conjunctors, since the general bounds F_G, F_{L1}, F_{L2} and F_{L3} are not *SI* conjunctors. The greatest SI conjunctor is now

$$F_{GSI}(l_i, l_j) = \begin{cases} l_n & \text{if } \max\{l_i, l_j\} = l_n, \\ l_{2n} & \text{otherwise,} \end{cases}$$

and one minimal element (not unique) is the conjunctor F_{LSI} introduced in Remark 9.

Not only for indifference, the propagation of the strong F-transitivity of the reciprocal preference relation to its associated strict preference relation is only fulfilled by some conjunctors. Next we characterize those conjunctors F.

Proposition 14 *Let F be a conjunctor on $\mathcal{L}_n^+$. F is a* SP *conjunctor if and only if one of the following conditions is satisfied:*

(1) $F(l_n, l_n) \leq l_n < F(l_{n+1}, l_{n+1})$.
(2) $F(l_{n+1}, l_{n+1}) = \max\{F(l_n, l_{n+1}), F(l_{n+1}, l_n)\} = l_n$.

PROOF. Let us suppose F is a SP conjunctor. Then, by Definition 12, $\mathcal{L}_n^{sF}(A) \neq \emptyset$. Thus, by Lemma 11, $F(l_n, l_n) \leq l_n$. Suppose F does not satisfy any of the two conditions in the statement. This is equivalent to suppose that $F(l_{n+1}, l_{n+1}) < l_n$ or $F(l_{n+1}, l_{n+1}) = l_n > \max\{F(l_n, l_{n+1}), F(l_{n+1}, l_n)\}$. Let $a, b, c \in A$ and $R \in \mathcal{L}_n(A)$ defined by $R(a, b) = R(b, c) = l_{n+1}$, $R(b, a) = R(c, b) = l_{n-1}$, $R(a, c) = R(c, a) = l_n$ and $R(d, e) = l_n$ for any $d, e \in A \setminus \{a, b, c\}$. It is easy to see that $R \in \mathcal{L}_n^{sF}(A)$. However, P_R is not transitive: $a \, P_R \, b$, $b \, P_R \, c$, but $a \, I_R \, c$. Hence, F is not a SP conjunctor, which is a contradiction.

Conversely, as both conditions imposed for F imply that $F(l_n, l_n) \leq l_n$ then, by Lemma 11, $\mathcal{L}_n^{sF}(A) \neq \emptyset$. Let now $R \in \mathcal{L}_n^{sF}(A)$ and $a, b, c \in A$ such that $a \, P_R \, b$ and $b \, P_R \, c$.

(1) If $F(l_{n+1}, l_{n+1}) > l_n$, then

$$R(a, c) \geq F(R(a, b), R(b, c)) \geq F(l_{n+1}, l_{n+1}) > l_n,$$

i.e., $a \, P_R \, c$.

(2) If $F(l_{n+1}, l_{n+1}) = \max\{F(l_n, l_{n+1}), F(l_{n+1}, l_n)\} = l_n$, then

$$R(a, c) \geq F(R(a, b), R(b, c)) \geq F(l_{n+1}, l_{n+1}) = l_n,$$

i.e., $a \, P_R \, c$ or $R(a, c) = l_n$. However, if $R(a, c) = l_n$, then we have $R(c, a) = l_n$ and

$$\begin{aligned} l_{n-1} &\geq \max\{R(c, b), R(b, a)\} \geq \\ &\geq \max\{F(R(c, a), R(a, b)), F(R(b, c), R(c, a))\} \geq \\ &\geq \max\{F(l_n, l_{n+1}), F(l_{n+1}, l_n)\} = l_n \end{aligned}$$

which is a contradiction.

Consequently, if F satisfies the required conditions, then P_R is transitive. Hence, F is a SP conjunctor. $\square$

Any t-norm and any t-conorm are *SP* conjunctors. Again F_{GSI} is the greatest one, but now F_{LSI} is not a minimal element, since

$$F_{LSP}(l_i, l_j) = \begin{cases} l_1 & \text{if } \max\{l_i, l_j\} = l_n, \\ l_{n+1} & \text{if } \min\{l_i, l_j\} = l_{2n}, \\ l_n & \text{otherwise,} \end{cases}$$

is such that $F_{LSP} < F_{LSI}$.

Proposition 15 *Every* SI *conjunctor is a* SP *conjunctor.*

PROOF. If F is a SI conjunctor, by Lemma 11, $F(l_n, l_n) \leq l_n$. By Proposition 13, two cases are possible:

(1) $F(l_n, l_n) = l_n$.
By the monotonicity of F, $F(l_{n+1}, l_{n+1}) \geq l_n$.
(a) If $F(l_{n+1}, l_{n+1}) > l_n$, then $F(l_n, l_n) \leq l_n < F(l_{n+1}, l_{n+1})$.
(b) If $F(l_{n+1}, l_{n+1}) = l_n$, then

$$l_n = F(l_n, l_n) \leq \max\{F(l_n, l_{n+1}), F(l_{n+1}, l_n)\} \leq F(l_{n+1}, l_{n+1}) = l_n.$$

Consequently, $F(l_{n+1}, l_{n+1}) = \max\{F(l_n, l_{n+1}), F(l_{n+1}, l_n)\} = l_n$.

(2) $F(l_n, l_n) < l_n < \max\{F(l_n, l_{n+1}), F(l_{n+1}, l_n)\}$.
Since $F(l_{n+1}, l_{n+1}) \geq \max\{F(l_n, l_{n+1}), F(l_{n+1}, l_n)\} > l_n$, we have

$$F(l_n, l_n) \leq l_n < F(l_{n+1}, l_{n+1}).$$

By Proposition 14, the proof is concluded. □

Remark 16 • *Since F_{LSP} is a* SP *conjunctor, but it is not a* SI *conjunctor, the converse of Proposition 15 is not true.*

• *According to the previous result, SI conjunctors not only ensure the transitivity of I_R, but the transitivity of P_R. Since the transitivity of $P_R \cup I_R$ is equivalent to the transitivity of both P_R and I_R (see, for instance, Roubens and Vincke [29, 3.3.1] or García-Lapresta and Rodríguez-Palmero [18, Theorem 1]), SI conjunctors ensure the transitivity of the weak preference relation $P_R \cup I_R$.*

According to García-Lapresta and Rodríguez-Palmero [18, Proposition 5], for every preference structure (P, I), the conditions $P \circ I \subseteq P$ and $I \circ P \subseteq P$ are equivalent. For this reason, we do not analyze SIP conjunctors, those which ensure $I_R \circ P_R \subseteq P_R$ for every $R \in \mathcal{L}_n^{sF}(A)$.

Proposition 17 *Let F be a conjunctor on $\mathcal{L}_n^+$. F is a* SPI *conjunctor if and only if F is a* SI *conjunctor.*

PROOF. First of all, $P_R \circ I_R \subseteq P_R$ implies that I_R is transitive (see, for instance, García-Lapresta and Rodríguez-Palmero [18, Proposition 5]). Then, every SPI conjunctor is always a SI conjunctor. Now suppose F is a SI conjunctor. Then, by Proposition 15, F is also a SP conjunctor. This implies that for any $R \in \mathcal{L}_n^{sF}(A)$, both P_R and I_R are transitive. This implies that $P_R \circ I_R \subseteq P_R$ (see, for instance, Roubens and Vincke [29, 3.3.1] or García-Lapresta and Rodríguez-Palmero [18, Theorem 1]). Consequently, F is a SPI conjunctor. $\square$

As we have characterized the family of conjunctors F such that

$$R \text{ is strongly } F\text{-transitive} \Longrightarrow \begin{cases} P_R \text{ is transitive,} \\ I_R \text{ is transitive,} \end{cases}$$

the following natural step is to study the converse implication, that is determine the strongest type of transitivity we can get for R from the transitivity of P_R and I_R.

Proposition 18 *Let F be a conjunctor on $\mathcal{L}_n^+$ and A a set of alternatives. F fulfills the following implication*

$$\left.\begin{array}{l} P_R \textit{ is transitive} \\ I_R \textit{ is transitive} \end{array}\right\} \Longrightarrow R \textit{ is strongly } F\textit{-transitive,}$$

for any preference structure (P, I) on A and for any $R \in \mathcal{L}_n(A)$ such that $P = P_R$ (and $I = I_R$), if, and only if, $F(l_{2n}, l_{2n}) = l_{n+1}$ and $F(l_n, l_n) \leq l_n$.

PROOF. Let us suppose F satisfies the implication. Let us denote the alternatives of A as $a_1, a_2, \ldots$ and let $R \in \mathcal{L}_n(A)$ be defined on A by

R	a_1	a_2	a_3	a_4	a_5	$\ldots$
a_1	l_n	l_{2n}	l_{n+1}	l_{2n}	l_{2n}	$\ldots$
a_2	l_0	l_n	l_{2n}	l_{2n}	l_{2n}	$\ldots$
a_3	l_{n-1}	l_0	l_n	l_{2n}	l_{2n}	$\ldots$
a_4	l_0	l_0	l_0	l_n	l_{2n}	$\ldots$
a_5	l_0	l_0	l_0	l_0	l_n	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Then,

P_R	a_1	a_2	a_3	a_4	a_5	$\ldots$
a_1	0	1	1	1	1	$\ldots$
a_2	0	0	1	1	1	$\ldots$
a_3	0	0	0	1	1	$\ldots$
a_4	0	0	0	0	1	$\ldots$
a_5	0	0	0	0	0	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

I_R	a_1	a_2	a_3	a_4	a_5	$\ldots$
a_1	1	0	0	0	0	$\ldots$
a_2	0	1	0	0	0	$\ldots$
a_3	0	0	1	0	0	$\ldots$
a_4	0	0	0	1	0	$\ldots$
a_5	0	0	0	0	1	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

It is immediate that P_R and I_R are transitive. However, if $F(l_{2n}, l_{2n}) > l_{n+1}$, R is not strongly F-transitive: $F(R(a_1, a_2), R(a_2, a_3)) = F(l_{2n}, l_{2n}) > l_{n+1} = R(a_1, a_3)$. If $F(l_n, l_n) > l_n$, then $\mathcal{L}_n^{sF}(A) = \emptyset$ and therefore, neither can R be strongly F-transitive. Both assumptions lead to a contradiction.

Conversely, let us assume F is a conjunctor satisfying $F(l_{2n}, l_{2n}) = l_{n+1}$, $F(l_n, l_n) \leq l_n$ and R is a reciprocal preference relation on A based on $\mathcal{L}_n$ with P_R and I_R transitive. Then also $P_R \cup I_R$ is transitive. For all $a, b, c \in A$ with $R(a, b), R(b, c) \in \mathcal{L}_n^+$, we have that $aP_R \cup I_R b$ and $bP_R \cup I_R c$ and this implies that $aP_R \cup I_R c$ or, equivalently, that $R(a, c) \geq l_n$. Then, we have two cases:

(1) If $R(a, c) \geq l_{n+1}$, then $R(a, c) \geq F(l_{2n}, l_{2n}) \geq F(R(a, b), R(b, c))$.
(2) If $R(a, c) = l_n$, then $aI_R c$. Given that $R(a, b), R(b, c) \in \mathcal{L}_n^+$, then (i) $aP_R b$ or $aI_R b$ and (ii) $bP_R c$ or $bI_R c$. Since P_R and I_R are transitive, we have that $aI_R b$ and $bI_R c$. Thus, $F(R(a, b), R(b, c)) = F(l_n, l_n) \leq l_n = R(a, c)$.

Consequently, R is strongly F-transitive. □

From this proposition we obtain that the strongest type of strong transitivity we can get for R from the transitivity for P_R and I_R is the F_{GSR}-transitivity where

$$F_{GSR}(l_i, l_j) = \begin{cases} l_n & \text{if } \max\{l_i, l_j\} = l_n, \\ l_{n+1} & \text{otherwise.} \end{cases}$$

From Propositions 13, 15 and 18, it is immediate to prove the following result.

Corollary 19 *Let F be a conjunctor on $\mathcal{L}_n^+$ and let R be a reciprocal preference relation on A based on $\mathcal{L}_n$. F fulfills the following equivalence*

$$R \text{ is strongly } F\text{-transitive} \Longleftrightarrow \begin{cases} P_R \text{ is transitive,} \\ I_R \text{ is transitive,} \end{cases}$$

if, and only if, one of the following conditions is fulfilled:

(1) $F(l_n, l_n) = l_n$ *and* $F(l_{2n}, l_{2n}) = l_{n+1}$.
(2) $F(l_n, l_n) < l_n$ *and* $\max\{F(l_n, l_{2n}), F(l_{2n}, l_n)\} = F(l_{2n}, l_{2n}) = l_{n+1}$.

Neither t-norms nor t-conorms fulfill the conditions imposed in Corollary 19, therefore they do not guarantee the equivalence. Only for some of the "smallest" conjunctors we have the equivalence (for instance, F_{L2}, F_{L3} and F_{GSR}). As Remark 4 hints at, there are many different reciprocal relations associated to a preference structure (P, I). Then it is logical that the family of conjunctors fulfilling the equivalence for any $R \in \mathcal{L}_n(A)$ is reduced.

Example 20 *If we continue with Example 3, we already know that R is strongly S_M-transitive and it is easy to prove that P_R and I_R are transitive. However, the reciprocal preference relation R' defined by*

$$R'(x, y) = \begin{cases} l_8 & \textit{if } x = a, y = b, \\ l_5 & \textit{if } x = a, y = c, \\ l_6 & \textit{if } x = b, y = c, \\ R(x, y) & \textit{otherwise,} \end{cases}$$

has associated the same strict preference and indifference relations of Example 3, but it is not S_M-transitive. In fact, it is even not T-transitive for any *t-norm T:*

$$R'(a,c) = l_5 < l_6 = T(l_8, l_6) = T(R'(a,b), R'(b,c)) < S_M(R'(a,b), R'(b,c)).$$

This is not a drawback since t-norms and t-conorms do not fulfill the conditions presented in Proposition 18. Therefore the converse implication cannot be ensured for those operators.

5 The weak case

Since the weak F-transitivity of a reciprocal preference relation R does not impose any condition for l_n, it will not be possible to obtain similar results to Propositions 13 and 17 in this case. That is, it is impossible to find a conjunctor F with a proper behavior in the propagation of the transitivity from any $R \in \mathcal{L}_n^{wF}(A)$ to any structure including conditions over I_R. The following remark stresses this statement, showing that it is not possible to ensure the propagation of the transitivity to I_R not only in general, but also for any particular conjunctor.

Remark 21 For every conjunctor F on $\mathcal{L}_n^+$ such that $F(l_n, l_n) \leq l_n$ and for any set of alternatives A, there exists a relation $R \in \mathcal{L}_n^{wF}(A)$ such that I_R is not transitive and, consequently, the compositions of P_R and I_R in both senses are not included in P_R. Let us fix three alternatives in A: a, b, c. Let R be defined on A as

R	a	b	c
a	l_n	l_n	l_{n+1}
b	l_n	l_n	l_n
c	l_{n-1}	l_n	l_n

and $R(x,y) = l_n$ for any $\{a,c\} \neq \{x,y\} \in A$. Obviously, $R \in \mathcal{L}_n^{wF}(A)$. However,

(1) I_R is not transitive: $a\, I_R\, b$, $b\, I_R\, c$, but $a\, P_R\, c$.
(2) $P_R \circ I_R \not\subseteq P_R$: $a\, P_R\, c$, $c\, I_R\, b$, but $a\, I_R\, b$.
(3) $I_R \circ P_R \not\subseteq P_R$: $b\, I_R\, a$, $a\, P_R\, c$, but $b\, I_R\, c$.

Even though weak transitivity cannot guarantee the transitivity of I_R, we will show that it is possible to characterize the propagation of the transitivity to P_R. Let us start with the definition of WP conjunctor, in the same way as we started in the strong case with Definition 12.

Definition 22 *Let F be a conjunctor on $\mathcal{L}_n^+$. We say that F is a* WP conjunctor *if P_R is transitive for every $R \in \mathcal{L}_n^{wF}(A)$.*

Proposition 23 *If F is a WP conjunctor on $\mathcal{L}_n^+$ and satisfies $F(l_n, l_n) \leq l_n$, then F is also a SP conjunctor.*

PROOF. By Lemma 11, $F(l_n, l_n) \leq l_n$ implies $\mathcal{L}_n^{sF}(A) \neq \emptyset$. If $R \in \mathcal{L}_n^{sF}(A)$, then $R \in \mathcal{L}_n^{wF}(A)$ and, consequently, P_R is transitive. □

Proposition 24 *Let F be a conjunctor on $\mathcal{L}_n^+$. F is a* WP *conjunctor if and only if $F(l_{n+1}, l_{n+1}) > l_n$.*

PROOF. First suppose that F is a WP conjunctor satisfying $F(l_{n+1}, l_{n+1}) \leq l_n$. Let $a, b, c \in A$ and $R \in \mathcal{L}_n(A)$ defined by $R(a, b) = R(b, c) = l_{n+1}$, $R(b, a) = R(c, b) = l_{n-1}$, $R(a, c) = R(c, a) = l_n$ and $R(d, e) = l_n$ for any $d, e \in A \setminus \{a, b, c\}$. It is easy to see that $R \in \mathcal{L}_n^{wF}(A)$. However, P_R is not transitive: $a\, P_R\, b$, $b\, P_R\, c$ and $a\, I_R\, c$. Hence, F is not a WP conjunctor.

Conversely, let $R \in \mathcal{L}_n^{wF}(A)$, $F(l_{n+1}, l_{n+1}) > l_n$ and $a, b, c \in A$ such that $a\, P_R\, b$ and $b\, P_R\, c$. Then, $R(a, c) \geq F(R(a, b), R(b, c)) \geq F(l_{n+1}, l_{n+1}) > l_n$, i.e., $a\, P_R\, c$. Hence, F is a WP conjunctor. □

Thus, the minimal elements of the class of *WP* conjunctors are:

$$F_{LWP1}(l_i, l_j) = \begin{cases} l_1 & \text{if } \min\{l_i, l_j\} = l_n, \\ l_{n+1} & \text{otherwise,} \end{cases}$$

$$F_{LWP2}(l_i, l_j) = \begin{cases} l_0 & \text{if } \min\{l_i, l_j\} = l_n \text{ and } l_i < l_{2n}, \\ l_{n+1} & \text{otherwise,} \end{cases}$$

$$F_{LWP3}(l_i, l_j) = \begin{cases} l_0 & \text{if } \min\{l_i, l_j\} = l_n \text{ and } l_j < l_{2n}, \\ l_{n+1} & \text{otherwise,} \end{cases}$$

and the maximum *WP* conjunctor is F_G.

In particular, any t-conorm is a *WP* conjuntor, since $S_M(l_{n+1}, l_{n+1}) = l_{n+1} > l_n$ and $S_M \leq S$, for any t-conorm S. However, this property is not fulfilled for any t-norm. Thus, T_M is a *WP* conjunctor, but the smallest t-norm (drastic t-norm) T_D defined by

$$T_D(l_i, l_j) = \begin{cases} l_n & \text{if } \max\{l_i, l_j\} < l_{2n}, \\ \min\{l_i, l_j\} & \text{otherwise,} \end{cases}$$

is not: $T_D(l_{n+1}, l_{n+1}) = l_n$.

Figure 1 contains the implications among the analyzed properties (Propositions 15, 17 and 23), for any conjunctor F such that $\mathcal{L}_n^{sF}(A) \neq \emptyset$.

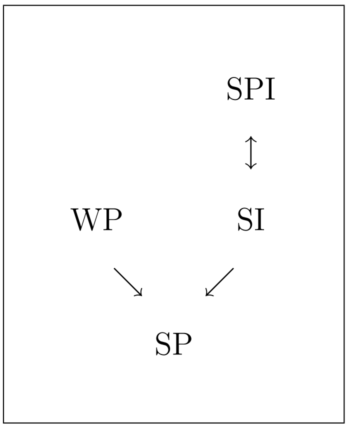

Fig. 1. Relationships among properties.

Remark 25 We note that the implications left in Figure 1 cannot be established. This holds just considering Remark 16 and the conjunctors T_D and F_G. T_D is *SI* and, therefore, *SP*, but it is not *WP*. On the other hand, F_G is *WP*, but it is not *SI*.

If $\mathcal{L}_n^{sF}(A) \neq \emptyset$, the requirement imposed to F in order to propagate the transitivity to P_R is stronger if the reciprocal preference relation is weakly F-transitive (Proposition 24) than if it is strongly F-transitive (Proposition 14).
Concerning the converse implication, in Proposition 18 we studied what happens when we want to guarantee the strong F-transitivity of the reciprocal relation. Now, although we only depart from the transitivity of P_R, the transitivity we want to obtain for R is also weaker. We show below that the class of conjunctors fulfilling the converse implication when dealing with weak transitivity is the same as the one obtained in Proposition 18.

Proposition 26 *Let F be a conjunctor on $\mathcal{L}_n^+$ and A a set of alternatives. F fulfills the following implication*

$$P \text{ is transitive } \Longrightarrow R \text{ is weakly } F\text{-transitive,}$$

for any P and for any $R \in \mathcal{L}_n(A)$ such that $P = P_R$, if, and only if, $F(l_{2n}, l_{2n}) = l_{n+1}$.

PROOF. Let us suppose F is a conjunctor, then $F(l_{2n}, l_{2n}) > l_n$. If F satisfies the implication and $F(l_{2n}, l_{2n}) > l_{n+1}$, let us denote the alternatives in A as $a_1, a_2, \ldots$ and let us define $R \in \mathcal{L}_n(A)$ as follows

R	a_1	a_2	a_3	a_4	a_5	$\ldots$
a_1	l_n	l_{2n}	l_{n+1}	l_{2n}	l_{2n}	$\ldots$
a_2	l_0	l_n	l_{2n}	l_{2n}	l_{2n}	$\ldots$
a_3	l_{n-1}	l_0	l_n	l_{2n}	l_{2n}	$\ldots$
a_4	l_0	l_0	l_0	l_n	l_{2n}	$\ldots$
a_5	l_0	l_0	l_0	l_0	l_n	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Then,

P_R	a_1	a_2	a_3	a_4	a_5	$\ldots$
a_1	0	1	1	1	1	$\ldots$
a_2	0	0	1	1	1	$\ldots$
a_3	0	0	0	1	1	$\ldots$
a_4	0	0	0	0	1	$\ldots$
a_5	0	0	0	0	0	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

I_R	a_1	a_2	a_3	a_4	a_5	$\ldots$
a_1	1	0	0	0	0	$\ldots$
a_2	0	1	0	0	0	$\ldots$
a_3	0	0	1	0	0	$\ldots$
a_4	0	0	0	1	0	$\ldots$
a_5	0	0	0	0	1	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

It is immediate that P_R and I_R are transitive. However, R is not weakly F-transitive: $F(R(a_1, a_2), R(a_2, a_3)) = F(l_{2n}, l_{2n}) > l_{n+1} = R(a_1, a_3)$. This is a contradiction.

Conversely, let us assume F is a conjunctor satisfying $F(l_{2n}, l_{2n}) = l_{n+1}$ and R is a reciprocal preference relation on A based on $\mathcal{L}_n$ with P_R transitive. For all $a, b, c \in A$ with $R(a, b), R(b, c) \in \mathcal{L}_n^{++}$, aP_Rb and bP_Rc. As P_R is transitive, then aP_Rc, that is, $R(a, c) \geq l_{n+1}$. Since $F(R(a, b), R(b, c)) \leq F(l_{2n}, l_{2n}) = l_{n+1}$, the proof is concluded. □

From Propositions 24 and 26, the equivalence is obtained for a very specific class of conjunctors.

Corollary 27 *Let F be a conjunctor on $\mathcal{L}_n^+$ and A a set of alternatives. F fulfills the equivalence*

$$R \text{ is weakly } F\text{-transitive} \iff P_R \text{ is transitive}$$

for all $R \in \mathcal{L}_n(A)$ if, and only if,

$$F(l_{n+1}, l_{n+1}) = F(l_{2n}, l_{2n}) = l_{n+1}.$$

Neither t-norms nor t-conorms fulfill the above requirements. Indeed, the class of conjunctor where the equivalence is fulfilled is really reduced. This is logical, as we commented before Example 20 for the strong case.

Example 28 *If we continue with Example 20, R is weakly S_M-transitive and R' is not weakly S_M-transitive. The weak S_M-transitivity of R implies the transitivity of P_R and then R and R' are weakly F_{GSR}-transitive.*

6 Concluding remarks

In preference modelling, it is usual to deal with conditions on rationality. This task is motivated by the necessity of ensuring consistent decisions, within the normative perspective. It is also motivated by the descriptive program, which provides several models of rationality for reflecting real human behavior (bounded rationality). In fact, there is a wide variety of rationality models in the literature which try to capture different notions of consistency.

Within preference modelling, the most usual framework is that of classical preferences, where individuals can only show preference or indifference. However, the fuzzy approach allows individuals to show different modalities of preference by means of numerical values within the unit interval. In order to deal with the human tendency to show preferences through linguistic terms rather than numerical values, the program *computing with words* (see Zadeh [36,37]) introduced preferences which take values in a set of linguistic labels.

In all the mentioned approaches, transitivity plays a crucial role for modelling consistency. However, there is a large class of transitivity conditions both in the fuzzy and in the linguistic frameworks. In this paper we have focused on reciprocal preferences. We have considered two classes of transitivity conditions, strong and weak, based on the very general class of conjunctors over some subsets of linguistic labels.

Our main task has been to analyze the propagation of transitivity from reciprocal preferences on an ordinal scale to the associated crisp strict preference and indifference relations and conversely. First of all, we established the natural requirements for an operator to define the transitivity in our context, as a generalization of the classical concept of transitivity in the case of crisp relations. Once these definitions were presented, we completely characterized the class of conjunctors which propagate the transitivity from ordinal reciprocal preference relations to the associated crisp indifference and preference relations. Moreover, we characterized the family of conjunctors fulfilling the converse propagation. Finally, we characterized those conjunctors which allow us to obtain the equivalence between transitivity in ordinal reciprocal and crisp preference relations. These results conclude the study both for strong and weak transitivity in the linguistic context. We have also shown the existing connections between all the obtained classes of conjunctors.

As future work, we would like to study the propagation of the transitivity when other ways of decomposing the reciprocal relation into crisp relations are considered.

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