The self-dual core and the anti-self-dual remainder of an aggregation operator

José Luis García-Lapresta [a] and Ricardo Alberto Marques Pereira [b]

[a] *Departamento de Economía Aplicada (Matemáticas), Universidad de Valladolid, Avenida Valle de Esgueva 6, 47011 Valladolid, Spain*

[b] *Dipartimento di Informatica e Studi Aziendali, Università degli Studi di Trento, Via Inama 5, TN 38100 Trento, Italy*

Abstract

In most decisional models based on pairwise comparison between alternatives, the reciprocity of the individual preference representations expresses a natural assumption of rationality. In those models self-dual aggregation operators play a central role, in so far as they preserve the reciprocity of the preference representations in the aggregation mechanism from individual to collective preferences. In this paper we propose a simple method by which one can associate a self-dual aggregation operator to any aggregation operator on the unit interval. The resulting aggregation operator is said to be the self-dual core of the original one, and inherits most of its properties. Our method constitutes thus a new characterization of self-duality, with some technical advantages relatively to the traditional symmetric sums method due to Silvert. In our framework, moreover, every aggregation operator can be written as a sum of a self-dual core and an anti-self-dual remainder which, in some cases, seems to give some indication on the dispersion of the variables. In order to illustrate the method proposed, we apply it to two important classes of continuous aggregation operators with the properties of idempotency, symmetry, and stability for translations: the OWA operators and the exponential quasiarithmetic means.

Keywords: aggregation operators, self-duality, symmetric sums, self-dual core, anti-self-dual remainder, stability for translations, OWA operators, exponential quasi-arithmetic means.

Email addresses: `lapresta@eco.uva.es` (José Luis García-Lapresta), `mp@economia.unitn.it` (Ricardo Alberto Marques Pereira).

1 Introduction

In a decisional context regarding a finite set of alternatives $\{a_1, \ldots, a_m\}$, the notion of graded pairwise comparison between alternatives is the basis for several important approaches to the question of modelling individual preferences. In spite of the mostly distinct nature of their frameworks, some of these models have significant formal and semantic analogies at the basic preference representation level. In this respect, we refer to the Skew-Symmetric Bilinear (SSB) model (see for instance Fishburn [16]) with real preference values, the Analytic Hierarchy Process (AHP) model (see for instance Saaty [34]) with positive preference values, and the Fuzzy Preference Relation (FPR) model (see for instance Nurmi [31]) with preference values in the $[0, 1]$ interval.

Notice that the fuzzy preference relation approach of the FPR model can be seen as a constrained version of the more general fuzzy preference structure approach (see De Baets and Fodor [8] for a thorough review). For this reason, the fuzzy preference relations in the FPR model are also called ipsodual or probabilistic relations (see for instance Doignon, Monjardet, Roubens and Vincke [10]).

In each approach to individual preference modelling, three hierarchical levels of rationality assumptions emerge. Consider for instance the FRP model, in which the degree of preference of one alternative a_k over another alternative a_l, with $k, l = 1, \ldots, m$, is expressed by the value $r_{kl} \in [0, 1]$ with the following interpretation: the value 0 expresses absolute preference of a_l over a_k, the value 1 expresses absolute preference of a_k over a_l, and the value 0.5 expresses indifference between the two alternatives.

Now, going back to the hierarchy of rationality assumptions: the first level rationality assumption requires the indifference between any alternative a_k and itself: $r_{kk} = 0.5$, for $k = 1, \ldots, m$. The second level rationality assumption, which implies the former one, requires the property of *reciprocity* in the pairwise comparison between any two alternatives a_k and a_l: $r_{lk} = 1 - r_{kl}$, for $k, l = 1, \ldots, m$. Finally, the third level rationality assumption, which implies the former two, requires the more complex property of consistency in the pairwise comparisons among any three alternatives.

The rationality assumptions regarding individual preferences play an important role in the question of constructing aggregated collective preferences from individual ones. If, for instance, one assumes that the individual fuzzy preference relations are reciprocal, it is then natural to require that the aggregated collective fuzzy preference relation be reciprocal as well. The aggregation rules having this property are called *reciprocal aggregation rules.*

In García-Lapresta and Llamazares [19] some results and references regarding the reciprocity of aggregation rules can be found. In particular it is proven that, under the general assumption of neutrality, the aggregation rules for n individual fuzzy preference relations are in one-to-one correspondence with the aggregation operators on n variables with values in the $[0,1]$ interval, that is functions of the form $A : [0,1]^n \longrightarrow [0,1]$. Moreover, the aggregation rules for fuzzy preference relations are reciprocal if and only if the corresponding aggregation operators on the $[0,1]$ interval are self-dual, that is $A(1-x_1,\ldots,1-x_n) = 1 - A(x_1,\ldots,x_n)$ for every $(x_1,\ldots,x_n) \in [0,1]^n$.

As far as aggregation is concerned, these results establish a clear correspondence between the fuzzy preference relation approach (based on pairwise comparisons between alternatives) and the standard evaluation approach (based on value assessment of single alternatives). Aggregation operators of the form $A : [0,1]^n \longrightarrow [0,1]$ are in fact typical of the latter approach, in which they are used for obtaining a collective value assessment of an alternative taking into account the individual ones.

An interesting implication of this correspondence is the fact that the reciprocity of the aggregation rules for fuzzy preference relations – ensuring that from reciprocal individual fuzzy preference relations one obtains a collective fuzzy preference relation which is also reciprocal – turns out to be equivalent to the self-duality of aggregation operators on the $[0,1]$ interval – ensuring that when all individuals reverse their preferences the aggregated collective preference is also reversed.

In general, however, aggregation operators are not self-dual. For this reason, we propose in this paper a simple method by which one can associate a self-dual aggregation operator to any aggregation operator on the $[0,1]$ interval. The resulting aggregation operator is said to be the self-dual core of the original one, and inherits most of its properties. Our method constitutes thus a new characterization of self-duality, with some technical advantages relatively to the traditional symmetric sums method due to Silvert [35].

In our framework, moreover, every aggregation operator decomposes as a sum of a self-dual core and an anti-self-dual remainder which, in some cases, seems to have an interesting interpretation as a measure of dispersion or dissensus among the individual preferences.

The paper is organized as follows. Section 2 is devoted to the presentation of basic notation and terminology, plus some standard facts on aggregation operators. Section 3 introduces the concepts of self-dual core and anti-self-dual remainder of an aggregation operator, establishing which properties are inherited in each case from the original aggregation operator. Particular emphasis is given to the properties of stability for translations (self-dual core) and invariance for translations (anti-self-dual remainder). Section 4 presents a similar analysis but now in the context of Silvert's symmetric sums approach. The study of which properties are inherited by the Silvert core (self-dual) and the Silvert remainder (not necessarily anti-self-dual) is described in detail, and several drawbacks of the traditional Silvert's approach are pointed out.

Finally, in Section 5 we apply our method to two important classes of continuous aggregation operators which are idempotent, symmetric, and stable for translations: the OWA operators introduced by Yager [36] and the quasiarithmetic means of exponential type. These are in fact the only quasiarithmetic means which are stable for translations, a well-known result due to Nagumo [30] (see also Kolmogoroff [24] for the classical characterization of quasiarithmetic means and Fodor and Roubens [18, chapter 5] for a good review).

The reason for considering OWA operators and exponential quasiarithmetic means is that these two classes of aggregation operators are relevant in every decision making context in which anonimity (symmetry), unanimity (idempotency), and uniformity (stability for translations) are required in the aggregation process. Notice that some OWA operators are actually self-dual (see García-Lapresta and Llamazares [20, Proposition 5]). In turn, the exponential quasiarithmetic means different to the arithmetic mean are never self-dual.

The self-dual core of an OWA operator is again an OWA operator. On the other hand, the self-dual core of an exponential quasiarithmetic mean is no longer decomposable, and therefore it is no longer a quasiarithmetic mean. An empirical study about individual and collective rationality using the self-dual cores of exponential quasiarithmetic means can be found in García-Lapresta and Meneses [22].

2 Aggregation operators

In this section we present notation and basic definitions regarding aggregation operators on $[0,1]^n$, with $n \in \mathbb{N}$ and $n \geq 2$ throughout the text.

Notation Points in $[0,1]^n$ will be denoted by means of boldface characters: $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\mathbf{1} = (1, \ldots, 1)$, $\mathbf{0} = (0, \ldots, 0)$. For $x \in [0,1]$, we have $x \cdot \mathbf{1} = (x, \ldots, x)$. Given $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$, by $\boldsymbol{x} \geq \boldsymbol{y}$ we mean $x_i \geq y_i$ for every $i \in \{1, \ldots, n\}$; by $\boldsymbol{x} > \boldsymbol{y}$ we mean $\boldsymbol{x} \geq \boldsymbol{y}$ and $\boldsymbol{x} \neq \boldsymbol{y}$. Moreover, $x_* = \min\{x_1, \ldots, x_n\}$ and $x^* = \max\{x_1, \ldots, x_n\}$. Given a permutation $\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$, with $\boldsymbol{x}_\sigma$ we denote $(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

We begin by defining standard properties of real functions on $[0,1]^n$.

Definition 1 *Let* $A : [0,1]^n \longrightarrow \mathbb{R}$ *be a function.*

(1) *A is* idempotent *if for every* $x \in [0,1]$*:* $A(x \cdot \mathbf{1}) = x$.

(2) *A is* symmetric *if for every permutation* $\sigma : \{1,\ldots,n\} \longrightarrow \{1,\ldots,n\}$ *and every* $\boldsymbol{x} \in [0,1]^n$*:* $A(\boldsymbol{x}_\sigma) = A(\boldsymbol{x})$.

(3) *A is* monotonic *if for all* $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$*:* $\boldsymbol{x} \geq \boldsymbol{y} \Rightarrow A(\boldsymbol{x}) \geq A(\boldsymbol{y})$.

(4) *A is* strictly monotonic *if for all* $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$*:* $\boldsymbol{x} > \boldsymbol{y} \Rightarrow A(\boldsymbol{x}) > A(\boldsymbol{y})$.

(5) *A is* compensative *if for every* $\boldsymbol{x} \in [0,1]^n$*:* $x_* \leq A(\boldsymbol{x}) \leq x^*$.

(6) *A is* self-dual *if for every* $\boldsymbol{x} \in [0,1]^n$*:* $A(\mathbf{1} - \boldsymbol{x}) = 1 - A(\boldsymbol{x})$.

(7) *A is* anti-self-dual *if for every* $\boldsymbol{x} \in [0,1]^n$*:* $A(\mathbf{1} - \boldsymbol{x}) = A(\boldsymbol{x})$.

(8) *A is* invariant for translations *if for all* $t \in [-1,1]$ *and* $\boldsymbol{x} \in [0,1]^n$*:* $A(\boldsymbol{x} + t \cdot \mathbf{1}) = A(\boldsymbol{x})$ *whenever* $\boldsymbol{x} + t \cdot \mathbf{1} \in [0,1]^n$.

(9) *A is* stable for translations *if for all* $t \in [-1,1]$ *and* $\boldsymbol{x} \in [0,1]^n$*:* $A(\boldsymbol{x} + t \cdot \mathbf{1}) = A(\boldsymbol{x}) + t$ *whenever* $\boldsymbol{x} + t \cdot \mathbf{1} \in [0,1]^n$.

Notice that a function A is simultaneously self-dual and anti-self-dual if and only if $A(\boldsymbol{x}) = 0.5$ for every $\boldsymbol{x} \in [0,1]^n$. Regarding the properties of invariance and stability for translations, we point out that the latter is also known as *shift invariance* (see Calvo, Kolesárová, Komorníková and Mesiar [6, p. 24] and Lázaro, Rückschlossová and Calvo [25]).

Definition 2 *A function* $A : [0,1]^n \longrightarrow [0,1]$ *is called an* n-ary aggregation operator *if it is monotonic and satisfies* $A(\mathbf{0}) = 0$ *and* $A(\mathbf{1}) = 1$*. For the sake of simplicity, the* n*-arity is omitted whenever it is clear from the context. An aggregation operator is said to be* strict *if it is strictly monotonic.*

Self-duality and stability for translations are important properties of aggregation operators. A well-known characterization of self-dual aggregation operators has been proposed in Silvert [35] (see also Calvo, Kolesárová, Komorníková and Mesiar [6, p. 32]). On the other hand, recent characterizations of aggregation operators which are stable for translations can be found in Marichal, Mathonet and Tousset [27] and Lázaro, Rückschlossová and Calvo [25].

In turn, anti-self-duality and invariance for translations are incompatible with the defining properties of aggregation operators, namely with the boundary conditions $A(\mathbf{0}) = 0$ and $A(\mathbf{1}) = 1$. Nevertheless, anti-self-duality and invariance for translations play an important role in this paper in so far as they are properties of important functions associated with aggregation operators, as we shall discuss later.

The following are standard facts concerning aggregation operators, proofs are straightforward.

Proposition 3 *Let* $A : [0,1]^n \longrightarrow [0,1]$ *be an aggregation operator.*

(1) A is idempotent if and only if A is compensative.

(2) If A is strict, then $A(\boldsymbol{x}) = 0$ *if only if* $\boldsymbol{x} = \mathbf{0}$*, and* $A(\boldsymbol{x}) = 1$ *if and only if* $\boldsymbol{x} = \mathbf{1}$*.*

(3) If A is stable for translations, then A is idempotent. ■

3 The self-dual core and the associated remainder

In this section we introduce the concepts of self-dual core and anti-self-dual remainder of an aggregation operator, establishing which properties are inherited in each case from the original aggregation operator. Particular emphasis is given to the properties of stability for translations (self-dual core) and invariance for translations (anti-self-dual remainder).

Definition 4 *Let* $A : [0,1]^n \longrightarrow [0,1]$ *be an aggregation operator.*

The aggregation operator $A^* : [0,1]^n \longrightarrow [0,1]$ *defined as*

$$A^*(\boldsymbol{x}) = 1 - A(\mathbf{1} - \boldsymbol{x})$$

is known as the dual *of the aggregation operator A.*

Clearly, $(A^*)^* = A$, which means that dualization is an *involution.*

An aggregation operator A is self-dual if and only if $A^* = A$. The properties of idempotency, symmetry, strict monotonicity, compensativeness, self-duality, anti-self-duality, invariance and stability for translations are all preserved by duality. The same holds for continuity.

Aggregation operators are not in general self-dual. However, a self-dual aggregation operator can be associated to any aggregation operator in a simple manner. The construction of the so-called *self-dual core* of an aggregation operator A is as follows.

Definition 5 *Let* $A : [0,1]^n \longrightarrow [0,1]$ *be an aggregation operator. The function* $\widehat{A} : [0,1]^n \longrightarrow [0,1]$ *defined by*

$$\widehat{A}(\boldsymbol{x}) = \frac{A(\boldsymbol{x}) + A^*(\boldsymbol{x})}{2} = \frac{A(\boldsymbol{x}) - A(\mathbf{1} - \boldsymbol{x}) + 1}{2}.$$

is called the core *of the aggregation operator A.*

Notice that $\widehat{A}$ is clearly an aggregation operator, verifying the boundary conditions $\widehat{A}(\mathbf{0}) = 0$, $\widehat{A}(\mathbf{1}) = 1$ and monotonicity. Moreover, $\widehat{A}$ is self-dual, since $\widehat{A}(\mathbf{1} - \boldsymbol{x}) = 1 - \widehat{A}(\boldsymbol{x})$ for every $\boldsymbol{x} \in [0,1]^n$. We say that $\widehat{A}$ is the *self-dual core* of the aggregation operator A.

The following result regards properties of the self-dual core, with straightforward proofs.

Proposition 6 *Let* $A : [0,1]^n \longrightarrow [0,1]$ *be an aggregation operator.*

(1) A is self-dual if and only if $\widehat{A}(\boldsymbol{x}) = A(\boldsymbol{x})$ for every $\boldsymbol{x} \in [0,1]^n$.

(2) Accordingly, $\widehat{\widehat{A}}(\boldsymbol{x}) = \widehat{A}(\boldsymbol{x})$ for every $\boldsymbol{x} \in [0,1]^n$. ■

The transformation from A to $\widehat{A}$ is therefore a *projection*.

The next result expresses the so-called dual symmetry of the self-dual core, the proof is straightforward.

Proposition 7 *Let* $A, A^* : [0,1]^n \longrightarrow [0,1]$ *be a dual pair of aggregation operators. The self-dual core $\widehat{A}$ of the aggregation operator A and the self-dual core $\widehat{A^*}$ of the dual aggregation operator A^* coincide, that is $\widehat{A} = \widehat{A^*}$.* ■

In other words, the self-dual core $\widehat{A}$ is in fact associated with the dual pair A, A^* and not just with A alone. We also conclude that the transformation from A to $\widehat{A}$ commutes with dualization, that is $(\widehat{A}\,)^* = \widehat{A^*}$.

The following result is an immediate consequence of Proposition 6.

Proposition 8 *An aggregation operator $A : [0,1]^n \longrightarrow [0,1]$ is self-dual if and only if there exists an aggregation operator $B : [0,1]^n \longrightarrow [0,1]$ such that $A = \widehat{B}$.* ■

Remark 9 If A and B are two aggregation operators such that $A = \widehat{B}$, we say that B *generates* the self-dual aggregation operator A. An interesting problem is to determine the equivalence class of generators of a self-dual aggregation operator $A : [0,1]^n \longrightarrow [0,1]$,

$$[A] = \{\text{aggregation operators } B : [0,1]^n \longrightarrow [0,1] \mid \widehat{B} = A\}.$$

Given a self-dual aggregation operator A, we have $\widehat{A} = A$ and thus $A \in [A]$. Since $A^* = A$, we also have $A^* \in [A]$. However, $[A]$ includes more generators. Take for instance $A : [0,1]^2 \longrightarrow [0,1]$ with $A(x_1, x_2) = (x_1 + x_2)/2$. Then, consider the class of aggregation operators $B_\alpha : [0,1]^2 \longrightarrow [0,1]$ defined as $B_\alpha(x_1, x_2) = (1/\alpha)\ln((e^{\alpha x_1} + e^{\alpha x_2})/2)$, with $\alpha \in \mathbb{R} \setminus \{0\}$. Since $\widehat{B}_\alpha(x_1, x_2) = (x_1 + x_2)/2 = A(x_1, x_2)$, it follows that $B_\alpha \in [A]$ for every $\alpha \in \mathbb{R} \setminus \{0\}$.

In general, given two aggregation operators $A, B : [0,1]^n \longrightarrow [0,1]$, we have that $\widehat{A} = \widehat{B}$ if and only if the function $A - B : [0,1]^n \longrightarrow \mathbb{R}$ is anti-self-dual, that is $A(\boldsymbol{x}) - B(\boldsymbol{x}) = A(\mathbf{1} - \boldsymbol{x}) - B(\mathbf{1} - \boldsymbol{x})$ for every $\boldsymbol{x} \in [0,1]^n$. In other words, the equivalence class of generators of a self-dual aggregation operator A is of the form

$$[A] = \{\text{aggregation operators } A + C : [0,1]^n \longrightarrow [0,1]\}$$

where $C : [0,1]^n \longrightarrow \mathbb{R}$ is any anti-self-dual function for which $A + C$ is still an aggregation operator. This requires, in particular, $C(\mathbf{0}) = C(\mathbf{1}) = 0$. For a more detailed treatment of this issue, see Maes, Saminger and De Baets [26]. Note however that no characterization is known of the equivalence class [A] of generating aggregation operators, although more can be said of the larger equivalence class of generating functions.

The self-dual core $\widehat{A}$ inherits from the aggregation operator A the properties of idempotency (hence, compensativeness), symmetry and strict monotonicity, whenever A has these properties. The same holds for continuity.

The following result concerns another relevant property of the self-dual core: it inherits from the original aggregation operator the property of stability for translations. The proof is straightforward.

Proposition 10 *If an aggregation operator $A : [0,1]^n \longrightarrow [0,1]$ is stable for translations, then its self-dual core $\widehat{A}$ is also stable for translations.* ■

Having defined the self-dual core of an aggregation operator, we now introduce the *anti-self-dual remainder* $\widetilde{A}$, which is simply the difference between the original aggregation operator A and its self-dual core $\widehat{A}$.

Definition 11 *Let $A : [0,1]^n \longrightarrow [0,1]$ be an aggregation operator. The function $\widetilde{A} : [0,1]^n \longrightarrow \mathbb{R}$ defined by $\widetilde{A}(\boldsymbol{x}) = A(\boldsymbol{x}) - \widehat{A}(\boldsymbol{x})$, that is*

$$\widetilde{A}(\boldsymbol{x}) = \frac{A(\boldsymbol{x}) - A^*(\boldsymbol{x})}{2} = \frac{A(\boldsymbol{x}) + A(\mathbf{1} - \boldsymbol{x}) - 1}{2},$$

is called the remainder *of the aggregation operator A.*

Notice that $\widetilde{A}$ is anti-self-dual, since $\widetilde{A}(\mathbf{1} - \boldsymbol{x}) = \widetilde{A}(\boldsymbol{x})$ for every $\boldsymbol{x} \in [0,1]^n$. We say that $\widetilde{A}$ is the *anti-self-dual remainder* of the aggregation operator A.

Clearly, $\widetilde{A}$ is not an aggregation operator. In particular, $\widetilde{A}(\mathbf{0}) = \widetilde{A}(\mathbf{1}) = 0$ violates the boundary conditions and implies that $\widetilde{A}$ is either non monotonic or everywhere null. Moreover, $-0.5 \leq \widetilde{A}(\boldsymbol{x}) \leq 0.5$ for every $\boldsymbol{x} \in [0,1]^n$.

The following result regards properties of the anti-self-dual remainder, with straightforward proofs.

Proposition 12 *Let* $A : [0,1]^n \longrightarrow [0,1]$ *be an aggregation operator.*

(1) A is self-dual if and only if $\widetilde{A}(\boldsymbol{x}) = 0$ *for every* $\boldsymbol{x} \in [0,1]^n$.

(2) Accordingly, $\widetilde{\widetilde{A}}(\boldsymbol{x}) = 0$ *for every* $\boldsymbol{x} \in [0,1]^n$. ■

The remainder $\widetilde{A}$ is symmetric, whenever the aggregation operator A has that property. The same holds for continuity.

Summarizing, every aggregation operator A decomposes additively $A = \widehat{A} + \widetilde{A}$ in two components: the self-dual core $\widehat{A}$ and the anti-self-dual remainder $\widetilde{A}$, where only $\widehat{A}$ is an aggregation operator. The so-called *dual decomposition* $A = \widehat{A} + \widetilde{A}$ clearly shows some analogy with other algebraic decompositions, such as that of square matrices and bilinear tensors into their symmetric and skew-symmetric components.

The following result concerns two more properties of the anti-self-dual remainder, with straightforward proofs based directly on the definition $\widetilde{A} = A - \widehat{A}$ and the corresponding properties of the self-dual core.

Proposition 13 *Let* $A : [0,1]^n \longrightarrow [0,1]$ *be an aggregation operator.*

(1) If A is idempotent, then $\widetilde{A}(x \cdot \mathbf{1}) = 0$ *for every* $x \in [0,1]$.

(2) If A is stable for translations, then $\widetilde{A}$ *is invariant for translations.* ■

These properties of the anti-self-dual remainder are suggestive. The first statement applies to the class of idempotent aggregation operators. In such case, self-dual cores are idempotent and therefore anti-self-dual remainders are null on the main diagonal. The second statement applies to the subclass of stable aggregation operators. In such case, self-dual cores are stable and therefore anti-self-dual remainders are invariant for translations. In other words, if the aggregation operator A is stable for translations, the value $\widetilde{A}(\boldsymbol{x})$ does not depend on the average value of the $\boldsymbol{x}$ coordinates, but only on their numerical deviations from that average value. These properties of the anti-self-dual remainder $\widetilde{A}$ suggest that it may give some indication on the dispersion of the $\boldsymbol{x}$ coordinates.

4 Symmetric sums and Silvert's approach

The analysis presented in this section is analogous to that presented in the previous one, but now we are in the context of Silvert's symmetric sums approach. The study of which properties are inherited by the Silvert core (self-dual) and the Silvert remainder (not necessarily anti-self-dual) is described in detail, and several drawbacks of the traditional Silvert's approach are pointed out.

Symmetric sums are self-dual aggregation operators, originally introduced by Silvert [35] in the case $n = 2$ within the context of his characterization of self-duality. Several authors have since considered symmetric sums: see for instance Dubois and Prade [11, p. 101] and Calvo, Kolesárova, Komorníková and Mesiar [6, p. 32], sometimes assuming further properties (eg. symmetry and continuity). In this paper, however, symmetric sums are simply taken to be self-dual aggregation operators, with no further assumptions.

Definition 14 *Let* $A : [0,1]^n \longrightarrow [0,1]$ *be an aggregation operator. The function* $\widehat{A}^S : [0,1]^n \longrightarrow [0,1]$ *defined as*

$$\widehat{A}^S(\boldsymbol{x}) = \begin{cases} \dfrac{A(\boldsymbol{x})}{A(\boldsymbol{x}) + A(\mathbf{1}-\boldsymbol{x})}\,, & \textit{if } A(\boldsymbol{x}) + A(\mathbf{1}-\boldsymbol{x}) \neq 0, \\ 0.5\,, & \textit{otherwise} \end{cases}$$

is called the Silvert core *of the aggregation operator* A*.*

Remark 15 Given an aggregation operator $A : [0,1]^n \longrightarrow [0,1]$, we have that $0 \leq A(\boldsymbol{x}) + A(\mathbf{1}-\boldsymbol{x}) \leq 2$ for each $\boldsymbol{x} \in [0,1]^n$. Particularly, $A(\boldsymbol{x}) + A(\mathbf{1}-\boldsymbol{x}) = 0$ if and only if $A(\boldsymbol{x}) = A(\mathbf{1}-\boldsymbol{x}) = 0$, and $A(\boldsymbol{x}) + A(\mathbf{1}-\boldsymbol{x}) = 2$ if and only if $A(\boldsymbol{x}) = A(\mathbf{1}-\boldsymbol{x}) = 1$. Consider the aggregation operator $A : [0,1]^2 \longrightarrow [0,1]$ defined as $A(x,y) = xy$; we have that $A(x,y) + A(1-x,1-y)$ takes the value 0 at $(1,0)$ and $(0,1)$. On the other hand, if $A : [0,1]^2 \longrightarrow [0,1]$ is the aggregation operator defined as $A(x,y) = 1-(1-x)(1-y)$, then $A(x,y) + A(1-x,1-y)$ takes the value 2 at $(1,0)$ and $(0,1)$. Finally, if the aggregation operator A is strict, then the inequalities are strict, that is $0 < A(\boldsymbol{x}) + A(\mathbf{1}-\boldsymbol{x}) < 2$ for every $\boldsymbol{x} \in [0,1]^n$.

The next result ensures that $\widehat{A}^S$ is an aggregation operator, verifying the boundary conditions and monotonicity.

Proposition 16 *Let* $A : [0,1]^n \longrightarrow [0,1]$ *be an aggregation operator. The Silvert core* $\widehat{A}^S$ *is an aggregation operator.*

Proof. Clearly, $\widehat{A}^S(\mathbf{0}) = 0$ and $\widehat{A}^S(\mathbf{1}) = 1$. In order to establish the monotonicity of $\widehat{A}^S$, take $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$ with $\boldsymbol{x} \geq \boldsymbol{y}$, which implies $A(\boldsymbol{x}) \geq A(\boldsymbol{y})$ and $A(\mathbf{1}-\boldsymbol{x}) \leq A(\mathbf{1}-\boldsymbol{y})$ due to the monotonicity of A. Then,

(1) If $A(\boldsymbol{x}) \geq A(\boldsymbol{y}) > 0$, we have $A(\mathbf{1}-\boldsymbol{x})/A(\boldsymbol{x}) \leq A(\mathbf{1}-\boldsymbol{y})/A(\boldsymbol{y})$ and therefore

$$\widehat{A}^S(\boldsymbol{x}) = \frac{1}{1 + A(\mathbf{1}-\boldsymbol{x})/A(\boldsymbol{x})} \geq \frac{1}{1 + A(\mathbf{1}-\boldsymbol{y})/A(\boldsymbol{y})} = \widehat{A}^S(\boldsymbol{y})\,.$$

(2) If $A(\boldsymbol{x}) > A(\boldsymbol{y}) = 0$, two cases may occur:

(a) If $A(\mathbf{1}-\boldsymbol{y})=0$, then $A(\mathbf{1}-\boldsymbol{x})=0$, in which case $\widehat{A}^S(\boldsymbol{x})=1$ and $\widehat{A}^S(\boldsymbol{y})=0.5$, so that $\widehat{A}^S(\boldsymbol{x})>\widehat{A}^S(\boldsymbol{y})$.

(b) If $A(\mathbf{1}-\boldsymbol{y})>0$, we have $\widehat{A}^S(\boldsymbol{y})=0$ and $\widehat{A}^S(\boldsymbol{x})>0$, so that $\widehat{A}^S(\boldsymbol{x})>\widehat{A}^S(\boldsymbol{y})$ as before.

(3) If $A(\boldsymbol{x})=A(\boldsymbol{y})=0$, three cases may occur:

(a) If $A(\mathbf{1}-\boldsymbol{y})\geq A(\mathbf{1}-\boldsymbol{x})>0$, then $\widehat{A}^S(\boldsymbol{x})=\widehat{A}^S(\boldsymbol{y})=0$.

(b) If $A(\mathbf{1}-\boldsymbol{y})>A(\mathbf{1}-\boldsymbol{x})=0$, then $\widehat{A}^S(\boldsymbol{x})=0.5>0=\widehat{A}^S(\boldsymbol{y})$.

(c) If $A(\mathbf{1}-\boldsymbol{y})=A(\mathbf{1}-\boldsymbol{x})=0$, then $\widehat{A}^S(\boldsymbol{x})=\widehat{A}^S(\boldsymbol{y})=0.5$. ■

Proposition 17 *Let $A:[0,1]^n\longrightarrow[0,1]$ be an aggregation operator. The Silvert core $\widehat{A}^S$ is self-dual.*

Proof. Let $\boldsymbol{x}\in[0,1]^n$. Two cases may occur:

(1) $A(\boldsymbol{x})+A(\mathbf{1}-\boldsymbol{x})\neq 0$, in which case we have

$$\widehat{A}^S(\mathbf{1}-\boldsymbol{x})=\frac{A(\mathbf{1}-\boldsymbol{x})}{A(\mathbf{1}-\boldsymbol{x})+A(\boldsymbol{x})}=1-\widehat{A}^S(\boldsymbol{x}).$$

(2) $A(\boldsymbol{x})+A(\mathbf{1}-\boldsymbol{x})=0$, that is $A(\boldsymbol{x})=A(\mathbf{1}-\boldsymbol{x})=0$, in which case we have both $\widehat{A}^S(\boldsymbol{x})=0.5$ and $\widehat{A}^S(\mathbf{1}-\boldsymbol{x})=0.5$ directly from the definition. Once more, $\widehat{A}^S(\mathbf{1}-\boldsymbol{x})=1-\widehat{A}^S(\boldsymbol{x})$. ■

In Silvert's seminal paper [35], the self-dual Silvert core of the aggregation operator A is called the *symmetric sum* generated by A.

The following result regards properties of the Silvert core, with straightforward proofs.

Proposition 18 *Let $A:[0,1]^n\longrightarrow[0,1]$ be an aggregation operator.*

(1) A is self-dual if and only if $\widehat{A}^S(\boldsymbol{x})=A(\boldsymbol{x})$ for every $\boldsymbol{x}\in[0,1]^n$.

(2) Accordingly, $\widehat{\widehat{A}^S}^S(\boldsymbol{x})=\widehat{A}^S(\boldsymbol{x})$ for every $\boldsymbol{x}\in[0,1]^n$. ■

Contrary to our approach, the Silvert core of an aggregation operator A does not in general coincide with the Silvert core of the dual aggregation operator A^*, as illustrated below. This lack of dual symmetry is one of the drawbacks of Silvert's approach.

Proposition 19 *For any aggregation operator $A:[0,1]^n\longrightarrow[0,1]$, we have that $\widehat{A}^S=\widehat{A^*}^S$ if and only if either $A(\mathbf{1}-\boldsymbol{x})=A(\boldsymbol{x})$ or $A(\mathbf{1}-\boldsymbol{x})=1-A(\boldsymbol{x})$ holds at each point $\boldsymbol{x}\in[0,1]^n$, independently from point to point.*

PROOF. Suppose first that $\widehat{A}^S(\boldsymbol{x}) = \widehat{A^*}^S(\boldsymbol{x})$ for every $\boldsymbol{x} \in [0,1]^n$.

If $A(\boldsymbol{x})+A(\mathbf{1}-\boldsymbol{x}) = 0$, then $A(\boldsymbol{x}) = A(\mathbf{1}-\boldsymbol{x}) = 0$, and if $A(\boldsymbol{x})+A(\mathbf{1}-\boldsymbol{x}) = 2$, then $A(\boldsymbol{x}) = A(\mathbf{1}-\boldsymbol{x}) = 1$. On the other hand, if $0 < A(\boldsymbol{x}) + A(\mathbf{1}-\boldsymbol{x}) < 2$, we have that

$$\widehat{A}^S(\boldsymbol{x}) = \frac{A(\boldsymbol{x})}{A(\boldsymbol{x}) + A(\mathbf{1}-\boldsymbol{x})} \quad \text{and} \quad \widehat{A^*}^S(\boldsymbol{x}) = \frac{1 - A(\mathbf{1}-\boldsymbol{x})}{2 - A(\boldsymbol{x}) - A(\mathbf{1}-\boldsymbol{x})} \quad .$$

Therefore, given $\widehat{A}^S(\boldsymbol{x}) = \widehat{A^*}^S(\boldsymbol{x})$, straightforward algebra leads to

$$\begin{aligned}
&A(\boldsymbol{x})(1 - A(\boldsymbol{x})) = A(\mathbf{1}-\boldsymbol{x})(1 - A(\mathbf{1}-\boldsymbol{x})) \\
&A(\boldsymbol{x})(1 - A(\boldsymbol{x}) - A(\mathbf{1}-\boldsymbol{x})) = A(\mathbf{1}-\boldsymbol{x})(1 - A(\boldsymbol{x}) - A(\mathbf{1}-\boldsymbol{x})) \\
&(A(\boldsymbol{x}) - A(\mathbf{1}-\boldsymbol{x}))(1 - A(\boldsymbol{x}) - A(\mathbf{1}-\boldsymbol{x})) = 0.
\end{aligned}$$

which means either $A(\mathbf{1}-\boldsymbol{x}) = A(\boldsymbol{x})$ or $A(\mathbf{1}-\boldsymbol{x}) = 1 - A(\boldsymbol{x})$ holds at each point $\boldsymbol{x} \in [0,1]^n$, independently from point to point.

Reciprocally, suppose that either $A(\mathbf{1}-\boldsymbol{x}) = A(\boldsymbol{x})$ or $A(\mathbf{1}-\boldsymbol{x}) = 1 - A(\boldsymbol{x})$ holds at each point $\boldsymbol{x} \in [0,1]^n$, independently from point to point.

If $A(\boldsymbol{x})+A(\mathbf{1}-\boldsymbol{x}) = 0$ or $A(\boldsymbol{x})+A(\mathbf{1}-\boldsymbol{x}) = 2$, then $\widehat{A}^S(\boldsymbol{x}) = \widehat{A^*}^S(\boldsymbol{x}) = 0.5$. On the other hand, if $0 < A(\boldsymbol{x}) + A(\mathbf{1}-\boldsymbol{x}) < 2$, we have that

(1) If $A(\mathbf{1}-\boldsymbol{x}) = A(\boldsymbol{x})$, then $\widehat{A}^S(\boldsymbol{x}) = 0.5 = \widehat{A^*}^S(\boldsymbol{x})$.
(2) If $A(\mathbf{1}-\boldsymbol{x}) = 1 - A(\boldsymbol{x})$, then $\widehat{A}^S(\boldsymbol{x}) = A(\boldsymbol{x}) = A^*(\boldsymbol{x}) = \widehat{A^*}^S(\boldsymbol{x})$. ■

Remark 20 In particular, the dual symmetry $\widehat{A}^S = \widehat{A^*}^S$ holds when the aggregation operator A is self-dual, that is $A(\mathbf{1}-\boldsymbol{x}) = 1 - A(\boldsymbol{x})$ for every $\boldsymbol{x} \in [0,1]^n$ (recall that no aggregation operator is anti-self-dual). In general, though, dual symmetry holds even if $A(\mathbf{1}-\boldsymbol{x}) = 1 - A(\boldsymbol{x})$ holds only for some points $\boldsymbol{x} \in [0,1]^n$, whereas $A(\mathbf{1}-\boldsymbol{x}) = A(\boldsymbol{x})$ holds elsewhere. As an example, consider the well-known conjunctive uninorm $A : [0,1]^2 \longrightarrow [0,1]$ defined as

$$A(x,y) = \begin{cases} \dfrac{xy}{xy + (1-x)(1-y)}\ , & \text{if } (x,y) \in [0,1]^2 \setminus \{(1,0),\ (0,1)\} \\ 0\ , & \text{if } (x,y) \in \{(1,0),\ (0,1)\} \end{cases}$$

and its disjunctive dual uninorm A^*, which differs from A only at the two points $(1,0)$ and $(0,1)$ where A^* takes the value 1. Since the denominator $xy + (1-x)(1-y)$ is non null except at $(1,0)$ and $(0,1)$, it follows that $A(\mathbf{1}-\boldsymbol{x}) = 1 - A(\boldsymbol{x})$ everywhere except at the two points $(1,0)$ and $(0,1)$, whereas $A(\mathbf{1}-\boldsymbol{x}) = A(\boldsymbol{x})$ at $(1,0)$ and $(0,1)$. The same applies to the dual aggregation operator A^*. Therefore, according to with Proposition 19, dual symmetry holds, $\widehat{A}^S = \widehat{A^*}^S$. Indeed, the Silvert cores $\widehat{A}^S$ and $\widehat{A^*}^S$ coincide everywhere with the original aggregation operators A and A^*, except at the two points $(1,0)$ and $(0,1)$ where $\widehat{A}^S$ and $\widehat{A^*}^S$ take the value 0.5.

The following result, analogous to Proposition 8, is an immediate consequence of Proposition 18.

Proposition 21 *An aggregation operator* $A : [0,1]^n \longrightarrow [0,1]$ *is self-dual if and only if there exists an aggregation operator* $B : [0,1]^n \longrightarrow [0,1]$ *such that* $A = \widehat{B}^S$. ■

The Silvert core $\widehat{A}^S$ inherits from the aggregation operator A the properties of idempotency, symmetry, strict monotonicity and compensativeness, whenever A has these properties. As far as strict monotonicity is concerned, we recall that when A is strict we have $A(\boldsymbol{x}) = 0$ only at $\boldsymbol{x} = \mathbf{0}$ and therefore $A(\boldsymbol{x}) + A(\mathbf{1}-\boldsymbol{x}) \neq 0$ for every $\boldsymbol{x} \in [0,1]^n$. In such case, the strict monotonicity proof for $\widehat{A}^S$ is a simplified version of the monotonicity proof in Proposition 16.

The question of continuity is another drawback of Silvert's approach. Indeed, continuity is not necessarily preserved by the Silvert core, as illustrated in the following remark.

Remark 22 At those points $\boldsymbol{x} \in [0,1]^n$ where $A(\boldsymbol{x}) + A(\mathbf{1}-\boldsymbol{x}) = 0$, and thus $A(\boldsymbol{x}) = A(\mathbf{1}-\boldsymbol{x}) = 0$, the Silvert core $\widehat{A}^S$ is not necessarily continuous. As was pointed out in Silvert [35], $A : [0,1]^2 \longrightarrow [0,1]$ defined as $A(x,y) = xy$ is a continuous aggregation operator whose Silvert core $\widehat{A}^S$ is continuous everywhere except at the two points $(1,0)$ and $(0,1)$. In fact,

$$\widehat{A}^S(x,y) = \begin{cases} \dfrac{xy}{xy + (1-x)(1-y)} \ , & \text{if } (x,y) \in [0,1]^2 \setminus \{(1,0),\, (0,1)\} \\ 0.5 \ , & \text{if } (x,y) \in \{(1,0),\, (0,1)\} \end{cases}$$

satisfies

$$\lim_{x \to 0} \widehat{A}^S(x, 1-x) = 0.5 \neq 1 = \lim_{x \to 0} \widehat{A}^S(x,1) \neq 0 = \lim_{y \to 1} \widehat{A}^S(0,y)$$

as well as

$$\lim_{y \to 0} \widehat{A}^S(1-y, y) = 0.5 \neq 1 = \lim_{y \to 0} \widehat{A}^S(1,y) \neq 0 = \lim_{x \to 1} \widehat{A}^S(x,0) \ .$$

On the other hand, continuity is preserved by the Silvert core in the case of strict aggregation operators. This is due to the fact that strict monotonicity implies $A(\boldsymbol{x}) + A(\mathbf{1} - \boldsymbol{x}) \neq 0$ for every $\boldsymbol{x} \in [0,1]^n$, as explained earlier (Remark 15).

Proposition 23 *Let $A : [0,1]^n \longrightarrow [0,1]$ be a strict aggregation operator. If A is continuous, then $\widehat{A}^S$ is also continuous.* ■

Contrary to our approach, stability for translations is not necessarily preserved by the Silvert core, which constitutes a further drawback of Silvert's approach. Consider for instance the aggregation operator $A : [0,1]^3 \longrightarrow [0,1]$ defined as $A(x_1, x_2, x_3) = \ln((e^{x_1} + e^{x_2} + e^{x_3})/3)$. Although A is stable for translations, its Silvert core $\widehat{A}^S$ is not stable for translations,

$$\widehat{A}^S(0.2{+}0.2, 0.6{+}0.2, 0.7{+}0.2) = 0.690192 \neq 0.699060 = \widehat{A}^S(0.2, 0.6, 0.7){+}0.2.$$

Having defined the Silvert core of an aggregation operator, we now introduce the *Silvert remainder* $\widetilde{A}^S$, which is simply the difference between the original aggregation operator A and its Silvert core $\widehat{A}^S$.

Definition 24 *Let $A : [0,1]^n \longrightarrow [0,1]$ be an aggregation operator. The function $\widetilde{A}^S : [0,1]^n \longrightarrow \mathbb{R}$ defined as $\widetilde{A}^S(\boldsymbol{x}) = A(\boldsymbol{x}) - \widehat{A}^S(\boldsymbol{x})$, that is*

$$\widetilde{A}^S(\boldsymbol{x}) = \begin{cases} A(\boldsymbol{x}) \, \dfrac{A(\boldsymbol{x}) + A(\mathbf{1} - \boldsymbol{x}) - 1}{A(\boldsymbol{x}) + A(\mathbf{1} - \boldsymbol{x})} \,, & \text{if } A(\boldsymbol{x}) + A(\mathbf{1} - \boldsymbol{x}) \neq 0, \\[2ex] -0.5 \,, & \text{otherwise}, \end{cases}$$

is called the Silvert remainder *of the aggregation operator A.*

Clearly, $\widetilde{A}^S$ is not an aggregation operator. In particular, $\widetilde{A}^S(\mathbf{0}) = \widetilde{A}^S(\mathbf{1}) = 0$ violates the boundary conditions and implies that $\widetilde{A}^S$ is either non monotonic or everywhere null. Moreover, $-1 \leq \widetilde{A}^S(\boldsymbol{x}) \leq 1$ for every $\boldsymbol{x} \in [0,1]^n$.

Contrary to our approach, the Silvert remainder $\widetilde{A}^S$ is not in general anti-self-dual. Consider for instance the aggregation operator $A : [0,1]^2 \longrightarrow [0,1]$ defined as $A(x_1, x_2) = \ln((e^{x_1} + e^{x_2})/2)$. It follows that

$$\widetilde{A}^S(1 - 0.2, 1 - 0.6) = 0.023689 \neq 0.016046 = \widetilde{A}^S(0.2, 0.6).$$

Proposition 25 *Let $A : [0,1]^n \longrightarrow [0,1]$ be an aggregation operator. The Silvert remainder $\widetilde{A}^S$ is anti-self-dual if and only if either $A(\mathbf{1} - \boldsymbol{x}) = A(\boldsymbol{x})$ or $A(\mathbf{1} - \boldsymbol{x}) = 1 - A(\boldsymbol{x})$ holds at each point $\boldsymbol{x} \in [0,1]^n$, independently from point to point.*

PROOF. Suppose first that $\widetilde{A}^S$ is anti-self-dual, i.e. $\widetilde{A}^S(\mathbf{1} - \boldsymbol{x}) = \widetilde{A}^S(\boldsymbol{x})$ for every $\boldsymbol{x} \in [0,1]^n$.

If $A(\boldsymbol{x})+A(\mathbf{1}-\boldsymbol{x})=0$, then $A(\boldsymbol{x})=A(\mathbf{1}-\boldsymbol{x})=0$. On the other hand, if $A(\boldsymbol{x})+A(\mathbf{1}-\boldsymbol{x})\neq 0$, we have that

$$\widetilde{A}^S(\boldsymbol{x})=A(\boldsymbol{x})\ \frac{A(\boldsymbol{x})+A(\mathbf{1}-\boldsymbol{x})-1}{A(\boldsymbol{x})+A(\mathbf{1}-\boldsymbol{x})}$$

and analogously

$$\widetilde{A}^S(\mathbf{1}-\boldsymbol{x})=A(\mathbf{1}-\boldsymbol{x})\ \frac{A(\boldsymbol{x})+A(\mathbf{1}-\boldsymbol{x})-1}{A(\boldsymbol{x})+A(\mathbf{1}-\boldsymbol{x})}\ .$$

Given that $\widetilde{A}^S(\mathbf{1}-\boldsymbol{x})=\widetilde{A}^S(\boldsymbol{x})$, straightforward algebra leads to

$$A(\boldsymbol{x})(1-A(\boldsymbol{x})-A(\mathbf{1}-\boldsymbol{x}))=A(\mathbf{1}-\boldsymbol{x})(1-A(\boldsymbol{x})-A(\mathbf{1}-\boldsymbol{x}))$$
$$(A(\boldsymbol{x})-A(\mathbf{1}-\boldsymbol{x}))(1-A(\boldsymbol{x})-A(\mathbf{1}-\boldsymbol{x}))=0$$

which means either $A(\mathbf{1}-\boldsymbol{x})=A(\boldsymbol{x})$ or $A(\mathbf{1}-\boldsymbol{x})=1-A(\boldsymbol{x})$ holds at each point $\boldsymbol{x}\in[0,1]^n$, independently from point to point.

Reciprocally, suppose that either $A(\mathbf{1}-\boldsymbol{x})=A(\boldsymbol{x})$ or $A(\mathbf{1}-\boldsymbol{x})=1-A(\boldsymbol{x})$ holds at each point $\boldsymbol{x}\in[0,1]^n$, independently from point to point.

If $A(\boldsymbol{x})+A(\mathbf{1}-\boldsymbol{x})=0$, then $\widetilde{A}^S(\boldsymbol{x})=\widetilde{A}^S(\mathbf{1}-\boldsymbol{x})=-0.5$. On the other hand, if $A(\boldsymbol{x})+A(\mathbf{1}-\boldsymbol{x})\neq 0$, we have that

(1) If $A(\mathbf{1}-\boldsymbol{x})=A(\boldsymbol{x})$, then $\widetilde{A}^S(\boldsymbol{x})=A(\boldsymbol{x})-0.5=\widetilde{A}^S(\mathbf{1}-\boldsymbol{x})$.

(2) If $A(\mathbf{1}-\boldsymbol{x})=1-A(\boldsymbol{x})$, then $\widetilde{A}^S(\boldsymbol{x})=0=\widetilde{A}^S(\mathbf{1}-\boldsymbol{x})$. ■

The following result regards properties of the Silvert remainder, with straightforward proofs.

Proposition 26 *Let* $A:[0,1]^n\longrightarrow[0,1]$ *be an aggregation operator.*

(1) A is self-dual if and only if $\widetilde{A}^S(\boldsymbol{x})=0$ *for every* $\boldsymbol{x}\in[0,1]^n$.

(2) Accordingly, $\widetilde{\widetilde{A}^S}^S(\boldsymbol{x})=0$ *for every* $\boldsymbol{x}\in[0,1]^n$. ■

The Silvert remainder $\widetilde{A}^S$ is symmetric, whenever the aggregation operator A has this property. As far as continuity is concerned, what has been said of the Silvert core in Remark 22 and Proposition 23 applies also to the Silvert remainder through its definition $\widetilde{A}^S=A-\widehat{A}^S$. Continuity, therefore, is not necessarily preserved by the Silvert remainder.

As before, the idempotency of A implies that $\widetilde{A}^S(x\cdot\mathbf{1})=0$ for every $x\in[0,1]$. However, contrary to our approach, the fact that A is stable for translations does not necessarily imply that the Silvert remainder is invariant for translations. Consider for instance the aggregation operator $A : [0,1]^2 \longrightarrow [0,1]$ defined as $A(x_1, x_2) = \ln((e^{x_1} + e^{x_2})/2)$. Even though A is stable for translations, its Silvert remainder $\widetilde{A}^S$ is not invariant for translations,

$$\widetilde{A}^S(0.2 + 0.2, 0.6 + 0.2) = 0.023689 \neq 0.016046 = \widetilde{A}^S(0.2, 0.6).$$

As a final remark, we note that Maes, Saminger and De Baets [26] provide a characterization of self-dual aggregation operators which generalizes those given by Silvert [35] (in this paper, Proposition 21) and García-Lapresta and Marques Pereira [21] (in this paper, Proposition 8). In their framework, a central role is played by a family of binary aggregation operators satisfying a form of twisted self-duality condition. Each operator in this family corresponds to a particular way of combining an aggregation operator A with its dual A^* for the construction of a self-dual aggregation operator.

As special cases of the general approach proposed in Maes, Saminger and De Baets [26], one obtains our construction, based on the arithmetic mean, and Silvert's construction, based on the symmetric sums formula. Moreover, it is established that our construction is in fact the only one which preserves stability under translations.

5 OWA operators

In order to illustrate our construction of the self-dual core (self-dual also in Silvert's approach) and the anti-self-dual remainder (not necessarily anti-self-dual in Silvert's approach) of a general aggregation operator, we now turn to examine two important classes of continuous aggregation operators which are idempotent (hence, compensative), symmetric, and stable for translations.

One of these two classes, the one which we consider in this section, is that of OWA operators, a class of aggregation operators for which the aggregated value is obtained as a weighted average of the ordered $\boldsymbol{x}$ coordinate values. The ordering scheme assures the symmetry property.

Definition 27 *Given a weighting vector* $\boldsymbol{w} = (w_1, \dots, w_n) \in [0,1]^n$ *satisfying* $\sum_{i=1}^n w_i = 1$*, the* OWA operator associated with $\boldsymbol{w}$ *is the aggregation operator* $A_w : [0,1]^n \longrightarrow [0,1]$ *defined as*

$$A_w(\boldsymbol{x}) = \sum_{i=1}^n w_i \, x_{\sigma(i)} \; ,$$

where σ *is a permutation of* $\{1, \dots, n\}$ *such that* $x_{\sigma(1)} \geq \cdots \geq x_{\sigma(n)}$.

Simple examples of OWA operators are

$$A_w(\boldsymbol{x}) = \begin{cases} \max\{x_1, \ldots, x_n\}\,, & \text{when } \ \boldsymbol{w} = (1, 0, \ldots, 0)\,, \\ \min\{x_1, \ldots, x_n\}\,, & \text{when } \ \boldsymbol{w} = (0, \ldots, 0, 1\,)\,, \\ \dfrac{x_1 + \cdots + x_n}{n}\,, & \text{when } \ \boldsymbol{w} = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})\,. \end{cases}$$

As mentioned above, OWA operators are continuous, idempotent (hence, compensative), symmetric, and stable for translations. Moreover, an OWA operator A_w is self-dual if and only if $w_{n+1-i} = w_i$ for every $i \in \{1, \ldots, n\}$, see García-Lapresta and Llamazares [20, Proposition 5].

In general, the self-dual core $\widehat{A}_w$ and the anti-self-dual remainder $\widetilde{A}_w$ of an OWA operator A_w can be written as

$$\widehat{A}_w(\boldsymbol{x}) = \sum_{i=1}^{n} \frac{w_i + w_{n-i+1}}{2}\, x_{\sigma(i)} \quad \text{and} \quad \widetilde{A}_w(\boldsymbol{x}) = \sum_{i=1}^{n} \frac{w_i - w_{n-i+1}}{2}\, x_{\sigma(i)}\,.$$

As we know, the self-dual core $\widehat{A}_w$ is an aggregation operator. Moreover, since $\sum_{i=1}^{n}(w_i + w_{n-i+1})/2 = 1$, the self-dual core $\widehat{A}_w$ is again an OWA operator, that is $\widehat{A}_w = A_{\widehat{w}}$ with $\widehat{w}_i = (w_i + w_{n-i+1})/2$ for every $i \in \{1, \ldots, n\}$. Notice that $\widehat{A}_w$ reduces to the arithmetic mean in the simple case $n = 2$, but not in higher dimensions.

On the other hand, the anti-self-dual remainder $\widetilde{A}_w$ is not an aggregation operator. Notice, in particular, that $\widetilde{A}_w(\mathbf{1}) = \sum_{i=1}^{n}(w_i - w_{n-i+1})/2 = 0$.

The self-dual core and the anti-self-dual remainder can be equivalently written as follows,

$$\widehat{A}_w(\boldsymbol{x}) = \sum_{i=1}^{n} w_i\, \frac{x_{\sigma(i)} + x_{\sigma(n-i+1)}}{2} \quad \text{and} \quad \widetilde{A}_w(\boldsymbol{x}) = \sum_{i=1}^{n} w_i\, \frac{x_{\sigma(i)} - x_{\sigma(n-i+1)}}{2}\,.$$

These expressions show clearly that the self-dual core is a weighted average of pairwise averages of $\boldsymbol{x}$ coordinates (*quasi-midranges*; see David and Nagaraja [7, p. 242]), whereas the anti-self-dual remainder is a weighted average of pairwise differences of $\boldsymbol{x}$ coordinates (*quasi-ranges*; see David and Nagaraja [7, p. 248]). The anti-self-dual remainder is therefore independent of the overall average of the coordinates of $\boldsymbol{x}$ and constitutes a form of dispersion measure. Moreover, it is straightforward to prove that $w_1 \geq \cdots \geq w_n$ implies $\widetilde{A}_w(\boldsymbol{x}) \geq 0$ and $w_1 \leq \cdots \leq w_n$ implies $\widetilde{A}_w(\boldsymbol{x}) \leq 0$.

6 Quasiarithmetic means

In this section we consider another class of continuous aggregation operators which are idempotent (hence, compensative), symmetric, and stable for translations: the exponential quasiarithmetic means. These constitute an important family in the general class of quasiarithmetic means, for which the aggregated value is obtained by first transforming the values of the $\boldsymbol{x}$ coordinates and then transforming back the arithmetic mean of the transformed values.

Different studies on quasiarithmetic means and related aggregation operators can be found in Aczél [1–3], Aczél and Alsina [4], Dujmovic [12,13], Dyckhoff and Pedrycz [15], Dyckhoff [14], Bullen, Mitrinović and Vasić [5], Ovchinnikov [32], Fodor and Roubens [18, chapter 5], Calvo, Kolesárova, Komorníková and Mesiar [6], and Grabisch, Marichal, Mesiar and Pap [23], among others.

Definition 28 *A function* $\varphi : [0,1] \longrightarrow [0,1]$ *is an* order automorphism *if it is bijective and increasing.*

If φ is an order automorphism, then φ is continuous, with $\varphi(0) = 0$, $\varphi(1) = 1$, and φ^{-1} is an order automorphism with the same characteristics, see García-Lapresta and Llamazares [19, pp. 684-685]. The same applies to the dual order automorphism $\varphi^* : [0,1] \longrightarrow [0,1]$, defined as $\varphi^*(x) = 1 - \varphi(1-x)$ for every $x \in [0,1]$. Finally, inversion and dualization commute, that is $(\varphi^*)^{-1} = (\varphi^{-1})^*$, since $x = 1 - \varphi(\varphi^{-1}(1-x)) = \varphi^*(1 - \varphi^{-1}(1-x))$ and therefore $(\varphi^{-1})^*(x) = 1 - (\varphi^{-1})(1-x)$ for every $x \in [0,1]$.

Definition 29 *Let* $A : [0,1]^n \longrightarrow [0,1]$ *be an aggregation operator. We say that* A *is a* quasiarithmetic mean *if there exists an order automorphism* φ *such that*

$$A(\boldsymbol{x}) = \varphi^{-1}\left(\frac{\varphi(x_1) + \cdots + \varphi(x_n)}{n}\right)$$

where φ *is said to generate the quasiarithmetic mean* A.

The generating order automorphism φ is actually unique, as in the following result due to Aczél [1, p. 396].

Proposition 30 *Let* A *and* B *be two quasiarithmetic means, with generating order automorphisms* φ *and* ψ*, respectively. The quasiarithmetic means* A*,* B *coincide if and only if generating order automorphisms* φ*,* ψ *coincide.* ∎

The classical result that follows is due to Kolmogoroff [24] and Nagumo [30] (see also Fodor and Roubens [18, pp. 112-114]). It is an axiomatic characterization of the class of quasiarithmetic means based on the notion of *decomposability* (see Fodor and Roubens [18, pp. 112-114] and Calvo, Kolesárova, Komorníková and Mesiar [6, p. 27]).

An aggregation operator is said to be *decomposable* if, for any given subset of variables, every variable in the subset can have its value replaced by the partial aggregated value of the subset without altering the overall aggregated value of the full set of variables. The formal definition requires the complete multi-dimensional structure of an aggregation operator $A = \{A^{(n)}, n \geq 1\}$ involving a sequence of n-ary aggregation operators, the first of which is conventionally taken to be the identity function (see Calvo, Kolesárova, Komorníková and Mesiar [6]).

Proposition 31 *An aggregation operator A is a quasiarithmetic mean if and only if it is continuous, idempotent, symmetric, strictly monotonic and decomposable.* ∎

The assumption of strict monotonicity has been discussed by De Finetti [9] and more recently by Fodor and Marichal [17]. A different axiomatic characterization of quasiarithmetic means, based on the notion of *separability*, can be found in Aczél and Alsina [4] (see also Fodor and Roubens [18, pp. 116-117]).

Proposition 32 *Let A be a quasiarithmetic mean, with generating order automorphism φ. The dual quasiarithmetic mean A^* is generated by the dual order automorphism φ^*.*

Proof. Due to the commutativity of inversion and dualization, we have that $(\varphi^*)^{-1}(x) = (\varphi^{-1})^*(x) = 1 - (\varphi^{-1})(1-x)$ for every $x \in [0,1]$. Therefore,

$$(\varphi^*)^{-1}\left(\frac{\varphi^*(x_1) + \cdots + \varphi^*(x_n)}{n}\right) =$$
$$= 1 - \varphi^{-1}\left(1 - \frac{\varphi^*(x_1) + \cdots + \varphi^*(x_n)}{n}\right) =$$
$$= 1 - \varphi^{-1}\left(\frac{(1-\varphi^*(x_1)) + \cdots + (1-\varphi^*(x_n))}{n}\right) =$$
$$= 1 - \varphi^{-1}\left(\frac{\varphi(1-x_1) + \cdots + \varphi(1-x_n)}{n}\right) =$$
$$= 1 - A(\mathbf{1} - \boldsymbol{x}) = A^*(\boldsymbol{x}) \text{ for every } \boldsymbol{x} \in [0,1]^n \,. \qquad \blacksquare$$

The next result is an immediate corollary of Propositions 30 and 32 (see also García-Lapresta and Llamazares [20, pp. 471-473]).

Proposition 33 *The quasiarithmetic mean generated by an order automorphism φ is self-dual if and only if φ itself is self-dual, that is $\varphi(1-x) = 1-\varphi(x)$ for every $x \in [0,1]$.* ∎

The next result is due to Nagumo [30].

Proposition 34 *The quasiarithmetic mean generated by an order automorphism* φ *is stable for translations if and only if* $\varphi(x) = (e^{\alpha x} - 1)/(e^{\alpha} - 1)$ *for some* $\alpha \neq 0$, *or* $\varphi(x) = x$ *for every* $x \in [0,1]$. ■

The quasiarithmetic means generated by such order automorphisms are thus the only ones to be stable for translations, and constitute the so-called exponential family of quasiarithmetic means,

$$(1) \quad A_\alpha(\boldsymbol{x}) = \frac{1}{\alpha} \ln \left(\frac{e^{\alpha x_1} + \cdots + e^{\alpha x_n}}{n} \right) \quad \text{and}$$

$$(2) \quad A_0(\boldsymbol{x}) = \frac{x_1 + \cdots + x_n}{n} \ , \quad \text{with } \alpha \neq 0 \text{ above.}$$

The corresponding self-dual cores are

$$(1) \quad \widehat{A}_\alpha(\boldsymbol{x}) = \frac{1}{2\alpha} \ln \left(\frac{e^{\alpha x_1} + \cdots + e^{\alpha x_n}}{e^{-\alpha x_1} + \cdots + e^{-\alpha x_n}} \right) \quad \text{and}$$

$$(2) \quad \widehat{A}_0(\boldsymbol{x}) = \frac{x_1 + \cdots + x_n}{n} \ , \quad \text{with } \alpha \neq 0 \text{ above,}$$

and the anti-self-dual remainders are

$$(1) \quad \widetilde{A}_\alpha(\boldsymbol{x}) = \frac{1}{2\alpha} \ln \left(\frac{(e^{-\alpha x_1} + \cdots + e^{-\alpha x_n})(e^{\alpha x_1} + \cdots + e^{\alpha x_n})}{n^2} \right)$$

$$(2) \quad \text{and} \ \ \widetilde{A}_0(\boldsymbol{x}) = 0 \ , \quad \text{again with } \alpha \neq 0 \text{ above.}$$

The quasiarithmetic mean A_α in (1) is stable for translations but not self-dual, since the exponential order automorphism which generates it is not self-dual. On the other hand, the self-dual core $\widehat{A}_\alpha$ has both properties, i.e. self-duality and stability for translations.

The following result presents the standard parametric limits of the exponential family of quasiarithmetic means, plus the corresponding parametric limits for the self-dual cores and the anti-self-dual remainders. The proof is by straightforward application of l'Hospital's rule.

Proposition 35 *Concerning the exponential quasiarithmetic means,*

$$A_\alpha(\boldsymbol{x}) = \frac{1}{\alpha}\ln\left(\frac{e^{\alpha x_1} + \cdots + e^{\alpha x_n}}{n}\right) \quad \textit{with} \quad \alpha \neq 0$$

the following statements hold,

(1) *(a)* $\lim_{\alpha\to\infty} A_\alpha(\boldsymbol{x}) = x^*$ *(b)* $\lim_{\alpha\to-\infty} A_\alpha(\boldsymbol{x}) = x_*$

(2) $\lim_{\alpha\to 0} A_\alpha(\boldsymbol{x}) = \dfrac{x_1 + \cdots + x_n}{n} = A_0(\boldsymbol{x})$

(3) *(a)* $\lim_{\alpha\to\infty} \widehat{A}_\alpha(\boldsymbol{x}) = \dfrac{x_* + x^*}{2}$ *(b)* $\lim_{\alpha\to-\infty} \widehat{A}_\alpha(\boldsymbol{x}) = \dfrac{x_* + x^*}{2}$

(4) $\lim_{\alpha\to 0} \widehat{A}_\alpha(\boldsymbol{x}) = \dfrac{x_1 + \cdots + x_n}{n} = A_0(\boldsymbol{x})$

(5) *(a)* $\lim_{\alpha\to\infty} \widetilde{A}_\alpha(\boldsymbol{x}) = \dfrac{x^* - x_*}{2}$ *(b)* $\lim_{\alpha\to-\infty} \widetilde{A}_\alpha(\boldsymbol{x}) = -\dfrac{x^* - x_*}{2}$

(6) $\lim_{\alpha\to 0} \widetilde{A}_\alpha(\boldsymbol{x}) = 0\,.$ ■

Remark 36 The self-dual core $\widehat{A}_\alpha$ reduces to the arithmetic mean in the simple case $n = 2$, but not in higher dimensions. Take for instance the case $n = 3$ with $\alpha = 1$,

$$A_1(x_1, x_2, x_3) = \ln\left(\frac{e^{x_1} + e^{x_2} + e^{x_3}}{3}\right)$$

and

$$\widehat{A}_1(x_1, x_2, x_3) = \frac{1}{2}\ln\left(\frac{e^{x_1} + e^{x_2} + e^{x_3}}{e^{-x_1} + e^{-x_2} + e^{-x_3}}\right).$$

At $(x_1, x_2, x_3) = (0, 0, 1)$, we get $A_1(0, 0, 1) = 0.452832$ and $\widehat{A}_1(0, 0, 1) = 0.344725$, whereas the arithmetic mean is $1/3 = 0.333333$.

Remark 37 The self-dual core of an aggregation operator inherits most of its properties, but not decomposability. In particular, the self-dual core of the exponential quasiarithmetic mean A_α in (1) is not decomposable and therefore $\widehat{A}_\alpha$ is not a quasiarithmetic mean, as illustrated below.

Consider the case $n = 3$ with $\alpha = 1$ as before. At the point $(x_1, x_2, x_3) = (0, 0, 1)$, we get $A_1(0, 0, 1) = 0.452832$ and $\widehat{A}_1(0, 0, 1) = 0.344725$ as before. On the other hand, the partial aggregated values of the subset $\{x_2, x_3\}$ are $A_1(0, 1) = 0.620114$ and $\widehat{A}_1(0, 1) = 0.5$. Using these values to check decomposability, we get $A_1(0, 0.620114, 0.620114) = 0.452832$ and $\widehat{A}_1(0, 0.5, 0.5) = 0.331822$. Notice that $A_1(0, 0, 1) = A_1(0, 0.620114, 0.620114)$, whereas the two values $\widehat{A}_1(0, 0, 1)$ and $\widehat{A}_1(0, 0.5, 0.5)$ are different. This illustrates the fact that

Table 1
The remainder of an exponential quasiarithmetic mean compared with the variance

Points $\boldsymbol{x}$	$\widetilde{A}_1(\boldsymbol{x})$	$\frac{1}{2}\mathrm{Var}(\boldsymbol{x})$	$\frac{1}{2}\mathrm{Var}(\boldsymbol{x})/\widetilde{A}_1(\boldsymbol{x})$
$(0.692, 0.701, 0.647, 0.224)$	0.019657	0.019703	1.002336
$(0.308, 0.949, 0.559, 0.446)$	0.028330	0.028462	1.004655
$(0.215, 0.408, 0.127, 0.501)$	0.011037	0.011071	1.003063
$(0.469, 0.926, 0.195, 0.539)$	0.033908	0.034102	1.005743
$(0.209, 0.704, 0.483, 0.817)$	0.026806	0.026966	1.005961
$(0.018, 0.796, 0.596, 0.115)$	0.051822	0.052621	1.015430
$(0.217, 0.041, 0.854, 0.377)$	0.045388	0.045741	1.007774
$(0.802, 0.207, 0.562, 0.796)$	0.029191	0.029355	1.005605
$(0.455, 0.547, 0.642, 0.464)$	0.002845	0.002847	1.000627
$(0.221, 0.203, 0.086, 0.329)$	0.003711	0.003713	1.000631

the self-dual core $\widehat{A}_1$ is not decomposable, even though the original quasiarithmetic mean A_1 is so. Therefore, by Proposition 31, $\widehat{A}_1$ is not a quasiarithmetic mean operator.

Finally, we go back to the subject of Proposition 13, namely the fact that the anti-self-dual remainder of an aggregation operator which is stable for translations seems to give some indication on the dispersion of the $\boldsymbol{x}$ coordinates. It is straightforward to show that the Taylor polynomial of degree 2 of the remainder $\widetilde{A}_\alpha$ centered at $\boldsymbol{x}_0 = \boldsymbol{0}$ coincides with $\frac{\alpha}{2}\mathrm{Var}$, where $\mathrm{Var}(\boldsymbol{x})$ denotes the standard variance of $\boldsymbol{x} \in [0,1]^n$,

$$\mathrm{Var}(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^{n}\left(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j\right)^2 = \frac{n-1}{n^2}\sum_{i=1}^{n} x_i^2 - \frac{2}{n^2}\sum_{i,j=1,i<j}^{n} x_i x_j \ .$$

This fact is illustrated in Table 6, where the two functions $\widetilde{A}_1$ and Var are compared with respect to 10 randomly generated points in $[0,1]^4$. We see that the values of $\widetilde{A}_1$ are in fact very close to those of $\frac{1}{2}\mathrm{Var}$, the minor differences being due to the higher order terms in the Taylor expansion of $\widetilde{A}_1$.

7 Concluding remarks

We have described a simple method by which one can associate a self-dual aggregation operator to any aggregation operator on the $[0, 1]$ interval. In our framework, every aggregation operator A can be written as a sum of a self-dual core $\widehat{A}$ and an anti-self-dual remainder $\widetilde{A}$, $A = \widehat{A} + \widetilde{A}$. The self-dual core is an aggregation operator and preserves most of the properties of the original one. The anti-self-dual remainder has a different nature and, in some cases, gives some indication on the dispersion of the variables.

Our method also constitutes a new characterization of self-duality, with some technical advantages relatively to the traditional symmetric sums method due to Silvert. A detailed comparison of the two methods has been performed and several drawbacks of Silvert's approach have been instantiated, as listed below.

In order to illustrate the method proposed, we have applied it to two important classes of continuous aggregation operators with the properties of idempotency (hence, compensativeness), symmetry, and stability for translations: the OWA operators and the exponential quasiarithmetic means. These two classes of aggregation operators are relevant in decision making contexts in which anonimity (symmetry), unanimity (idempotency), and uniformity (stability for translations) are required in the aggregation process.

The case of the exponential quasiarithmetic means has provided interesting results, particularly the fact that the anti-self-dual remainder apparently acts as a measure of dispersion. The question is now being further investigated.

Finally, in Marques Pereira and Ribeiro [28] and Ribeiro and Marques Pereira [33] a different family of aggregation operators, the exponential mixtures, is derived from the exponential family of quasiarithmetic means. The exponential mixtures are continuous, idempotent (hence, compensative), symmetric, and stable for translations, just as the two classes of aggregation operators considered in this paper. Therefore, we believe that the application of our method may lead to interesting results also in the exponential mixtures case.

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